

Exact Small Time Equivalent for the density of the Circular Langevin Diffusion

J. Franchi

IRMA, Université de Strasbourg et CNRS,
7 rue René Descartes, 67084 Strasbourg, France

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Abstract

A small time equivalent of the density is obtained for the circular analogue of the Langevin diffusion, which is strictly hypoelliptic (and non-Gaussian), hence of a different nature as the known sub-Riemannian case. The singular case, analogous to the case of conjugate points (the cut-locus problem) in the sub-Riemannian framework, is totally handled too, though much more difficultly.

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1 Introduction

The problem of estimating the heat kernel, or the density of a diffusion, particularly as time goes to zero, has been extensively studied for a long time, firstly in the elliptic case, and then largely solved and understood in the sub-Riemannian case too. Let us mention only the articles [V], [A], [BA1], [BA2], [L], and the existence of other works on that subject by Azencott, Molchanov and Bismut, quoted in [BA1].

To summary roughly, a very classical question addresses the asymptotic behavior (as $s \searrow 0$) of the density $p_s(x, y)$ of the diffusion (x_s) solving a Stratonovich stochastic differential equation

$$x_s = x + \sum_{j=1}^k \int_0^s V_j(x_\tau) \circ dW_\tau^j + \int_0^s V_0(x_\tau) d\tau,$$

where the smooth vector fields V_j are supposed to satisfy a Hörmander condition.

The elliptic case being very well understood for a long time ([V], [A]), the studies focussed then on the sub-elliptic case, that is to say, when the strong Hörmander condition (that the Lie algebra generated by the fields V_1, \dots, V_k has maximal rank everywhere) is fulfilled. In that case these fields generate a sub-Riemannian distance $d(x, y)$, defined as in control theory, by considering only C^1 paths whose tangent vectors are spanned by them. Then the wanted asymptotic expansion tends to have the following Gaussian-like form :

$$p_s(x, y) = s^{-d/2} \exp(-d(x, y)^2/(2s)) \left(\sum_{\ell=0}^n \gamma_\ell(x, y) s^\ell + \mathcal{O}(s^{n+1}) \right) \quad (1)$$

for any $n \in \mathbb{N}^*$, with smooth γ_ℓ 's and $\gamma_0 > 0$, provided x, y are not conjugate points (and uniformly within any compact set which does not intersect the cut-locus). See in particular ([BA1], théorème 3.1). Note that the condition of remaining outside the cut-locus is here necessary, as showed in particular by [BA2].

The methods used to get this or a similar result have been of different nature. In [BA1], G. Ben Arous proceeds by expanding the flow associated to the diffusion (in this direction, see also [Ca]) and using a Laplace method applied to the Fourier transform of x_s , then inverted by means of Malliavin's calculus (with a deterministic Malliavin matrix).

The strictly hypoelliptic case, i.e., when only the weak Hörmander condition (requiring the use of the drift vector field V_0 to recover the full tangent space) is fulfilled, remains

much more problematic, and then rarely addressed. There is a priori no longer any reason that in such case the asymptotic behavior of $p_s(x, y)$ remains of the Gaussian-like type (1), all the less as a natural candidate for replacing the sub-Riemannian distance $d(x, y)$ is missing. Indeed this already fails for the mere (Gaussian) Langevin process $(\omega_s, \int_0^s \omega_\tau d\tau)$: the missing distance is replaced by a time-dependent distance $d_s(x, y)$ which presents some degeneracy in one direction, namely $d(x, y)^2/(2s)$ is replaced by

$$\frac{6}{s^3} \left| (x - y) - \frac{s}{2} (\dot{x} - \dot{y}) \right|^2 + \frac{1}{2s} |\dot{x} - \dot{y}|^2 = \frac{1}{2s} \left(|\dot{x} - \dot{y}|^2 + \frac{12}{s^2} \left| (x - y) - \frac{s}{2} (\dot{x} - \dot{y}) \right|^2 \right).$$

See also [DM] for a more involved (non-curved, strictly hypoelliptic, perturbed) case where Langevin-like estimates hold (without precise asymptotics), roughly having the following Li-Yau-like form :

$$C^{-1} s^{-N} e^{-C d_s(x_s, y)^2} \leq p_s(x, y) \leq C s^{-N} e^{-C^{-1} d_s(x_s, y)^2}, \quad \text{for } 0 < s < s_0. \quad (2)$$

In [F] was computed the small time asymptotics of a toy model, namely a diffusion in the second Wiener chaos, simplest case after the Gaussian setting, in the specific off-diagonal regime of a dominant normalized Gaussian contribution. The exponential term appeared as given by the same time-dependent distance as in the Langevin case, the strictly second chaos coordinate appearing only in the off-exponent term, as a perturbative contribution.

A stronger interest lies on a significant strictly hypoelliptic diffusion, namely the relativistic diffusion, first constructed in Minkowski's space (see [Du], [F-LJ2]). It makes sense on a generic smooth Lorentzian manifold as well, see [F-LJ1]. In the simplest case of Minkowski's space $\mathbb{R}^{1,d}$, it consists in the pair $(\xi_s, \dot{\xi}_s) \in \mathbb{R}^{1,d} \times \mathbb{H}^d$ (parametrized by its proper time s , and analogous to a Langevin process), where the velocity $(\dot{\xi}_s)$ is a hyperbolic Brownian motion. Note that even there, a curvature constraint must be taken into account, namely that of the mass shell \mathbb{H}^d , at the heart of this framework.

This (Dudley) relativistic diffusion, even restricted to 3 dimensions which already contain the essence of the difficulty, constitutes a significant example, altogether explicit, physical and not too much complicated, allowing a priori to progress towards the understanding of a more generic, but less accessible to begin with, strictly hypoelliptic case. However even this apparently simple example is not easy at all to analyze, regarding the small time asymptotics.

The present work handles a near example, but in which the curvature is non-negative and the fiber is compact, namely $\mathbb{R}^2 \times \mathbb{S}^1 \equiv T_+^1 \mathbb{R}^2$, endowed with its natural strictly hypoelliptic diffusion $(\xi_s, \dot{\xi}_s)$, analogue of both the Langevin and the Dudley diffusions, we name ‘‘circular Langevin diffusion’’. In this setting we obtain the exact small time ($\varepsilon \searrow 0$) equivalent for the density (heat kernel) $p_\varepsilon(x_0; x)$. To the author's knowledge, this is the first example of a complete result of this type in a strictly hypoelliptic framework. It reveals a different nature from the known sub-elliptic framework, see Section 2 below.

The present work was strongly influenced by the beautiful article [BA1], which decisively handled the off-cut locus generic sub-Riemannian framework, as far as the small-time asymptotics of the heat kernel is considered. Thus the strategy adopted below roughly resembles the strategy followed by G. Ben Arous, at least for the non-degenerate case. However the

present purpose is to deal with a strictly hypoelliptic situation, to which [BA1] does not apply, and to the best of the author's knowledge, remained unsolved. A main obstacle to handle a strictly hypoelliptic framework is the lack of sub-Riemannian distance. For that reason, the strategy used here only partially follows the method of [BA1], and stresses on the Brownian bridge instead of the Brownian motion. Another sensitive though more technical reason is that the Malliavin matrix which will intervene below is no longer deterministic as in [BA1], so that we shall resort to the treatment of infinite-dimensional oscillatory integrals (as in [T2], up to a modification).

As the author is not yet ready to handle a generic strictly hypoelliptic framework, and actually doubts that a generic unified treatment (or even, unified generic result) be possible (different strictly hypoelliptic frameworks could produce different types of results; the present one already differs notably from the classical Gaussian Langevin case, which even ignores degenerate cases), the focus is here put on a simple first example, which allows to concentrate on the heart of a method to get beyond the sub-Riemannian framework. In this way some difficulties handled in [BA1] don't have their counterpart here, in particular the asymptotic development is here limited to the first order term (i.e., only provides an equivalent as time goes to 0). Another simplification is due to the boundedness of the drift vector field V_0 , which does not hold in the relativistic (Dudley) case. It is not clear whether the latter could be handled similarly as is done here. On the contrary, the choice of considering a 3-dimensional case is unessential, but avoids even heavier notation and computations which higher dimensions would cause. Otherwise, focussing on the present (relatively) simple example allows to also handle the case of conjugate points; which is delicate, even in the sub-Riemannian framework: the cut locus constitutes a real difficulty in that matter, see for example [BA2], and its case does not seem to be generically solved in a sub-elliptic framework. Furthermore the choice of a relatively simple particular framework allows to fully explicit all coefficients of the wanted equivalents, which will likely be out of reach in an even slightly more generic framework.

The content is organized as follows.

In Section 2 are described the strictly hypoelliptic diffusion under consideration and the central result: Theorem 2.1.

Section 3 develops the leading strategy of the proof, which resembles that of [BA1], with some difference: whereas two main tools are as in [BA1], namely a Fourier-Parseval expression for the density of the heat kernel under consideration and the Girsanov transform, the Brownian component is here pinned, and the non-existing geodesic tube of [BA1] is here replaced by some partially pinned "pseudo-geodesic", which happens here to be directed by a mere segment. This leads to a somewhat simplified expression for the density kernel, which however contains an oscillatory integral which is not directly computable.

In Section 4 is analyzed the infinite-dimensional oscillatory integral which results from the preceding. As in [BA1] it is necessary to resort to Malliavin calculus, but in a more involved way since the Malliavin matrix is here no longer deterministic. A key argument will rely on the analysis of the decay of such infinite-dimensional oscillatory integrals, as performed in particular in [T2]. This requires technical estimates. Another key argument is an estimation

of a variance from below. As a consequence, a computable expression of the density is deduced in Section 5, in the non-degenerate case $w \neq 0$.

Section 6 is devoted to the singular case $w = 0$, which is much more delicate, and is analogous to the study at a sort of cut-locus, relating to some absent or hidden metric. In this section is performed a reduction to a simpler oscillatory integral, in a way resembling the non-degenerate case, but with a more delicate estimation of the variance.

In Section 7 the Fourier-Laplace transform of somme quadratic Brownian-bridge functional is computed, which is needed to solve the singular case $w = 0$.

Finally in the last section 8 is analyzed the delicate finite-dimensional oscillatory integral, which ultimately yields the wanted asymptotics of the singular case $w = 0$; which can have two sensibly different natures, depending on the target-point.

2 The circular Langevin diffusion and the main result

The circular Langevin diffusion reads $x_s = (\omega_s, y_s, z_s)$, with a standard real Brownian motion (ω_s) and

$$y_s := \int_0^s \cos \omega_\tau d\tau, \quad z_s := \int_0^s \sin \omega_\tau d\tau.$$

We have here parametrized $T_+^1 \mathbb{R}^2 \equiv \mathbb{S}^1 \times \mathbb{R}^2$ by $(\omega, y, z) \in \mathbb{R}^3$. A typical trajectory is depicted in Figure 1.

Consider the scaled diffusion $s \mapsto x_s^\varepsilon = \left(\sqrt{\varepsilon} \omega_s, \varepsilon \int_0^s \cos[\sqrt{\varepsilon} \omega_\tau] d\tau, \varepsilon \int_0^s \sin[\sqrt{\varepsilon} \omega_\tau] d\tau \right)$,

which has the same law as $[s \mapsto x_{\varepsilon s}^1 = x_{\varepsilon s}]$ and satisfies the stochastic differential equation

$$dx_s^\varepsilon = \sqrt{\varepsilon} V_1(x_s^\varepsilon) d\omega_s + \varepsilon V_0(x_s^\varepsilon) ds \quad (\text{analogous to (2.1) in [BA1]}),$$

with $V_1 = \partial_\omega$, $V_0 = \cos \omega \partial_y + \sin \omega \partial_z = [V_1, [V_1, V_0]]$, $[V_0, V_1] = \sin \omega \partial_y - \cos \omega \partial_z$,

which span $T\mathbb{R}^3$ at any point. Note that the vector fields V_j and their derivatives are here bounded. The Hörmander hypoellipticity criterion ensures the existence of a smooth density $p_\varepsilon(\cdot, \cdot)$ with respect to the Lebesgue measure for the relativistic diffusion x_ε . The density of x_1^ε is p_ε as well. We are interested in small times ε . According to Remark 2.2 below, by homogeneity we can restrict to the starting point 0.

We thus fix $(w, y, z) \in \mathbb{R}^3$, and look for the equivalent as time $\varepsilon \searrow 0$ of the generic value $p_\varepsilon(0; (w, y, z))$ of the density of x_1^ε . The idea is that $\sqrt{\varepsilon} \omega$ should concentrate around the linear $s \mapsto ws$, at least in the non-degenerate case $w \neq 0$ which we shall consider first. This non-degeneracy condition happens to be necessary in particular in the proof of Proposition 4.3.1 below, and, seen from the origin $(0, 0, 0)$, seems to correspond to an off-cut locus regime, whereas cut locus problems are significant, see [BA2]. This lets understand why the singular case $w = 0$ is sensibly more delicate, as appears in our result, Theorem 2.1 below.

The result to which this article is devoted is the following.

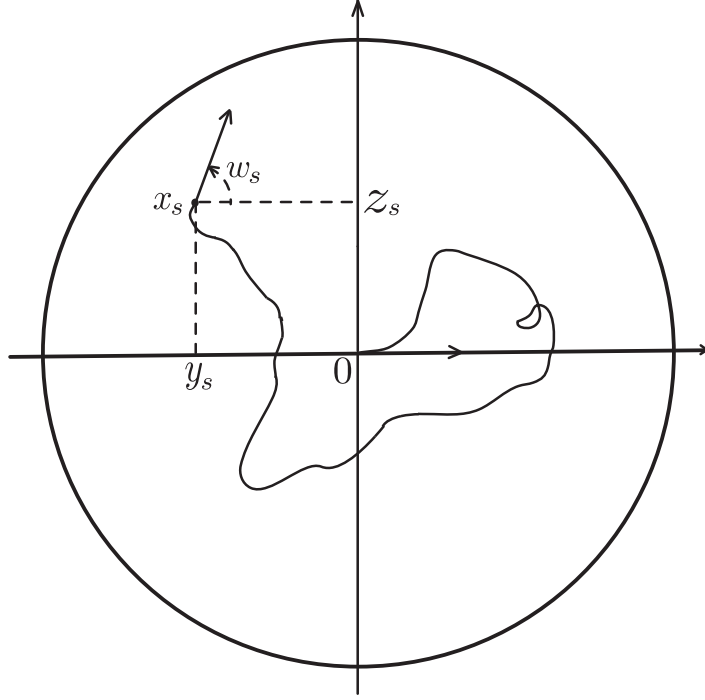


Figure 1: A circular Langevin trajectory from $0 \equiv (0, 0, 0)$ to $x_s = (w_s, y_s, z_s)$

Theorem 2.1 Consider a standard real Brownian motion (ω_s) , the circular Langevin diffusion $x_s = \left(\omega_s, \int_0^s \cos \omega_\tau d\tau, \int_0^s \sin \omega_\tau d\tau \right)_{s \geq 0}$, and denote by $p_s(0; \cdot)$ the density of the law of x_s started from $0 \equiv (0, 0, 0)$. It admits the following equivalents as $\varepsilon \searrow 0$.

(i) Non-degenerate case $w \neq 0$. For any $(w, y, z) \in \mathbb{R}^* \times \mathbb{R}^2$ we have :

$$p_\varepsilon(0; (w, y, z)) = \frac{(2 + o(1)) w^2}{\pi \varepsilon^3 \sqrt{2\pi \varepsilon \Delta(w)}} \times \exp \left[-\frac{w^2}{\varepsilon} \times \left(\frac{1}{2} + \frac{\psi(w, y/\varepsilon, z/\varepsilon)}{\Delta(w)} \right) \right],$$

with, to express the energy minimizing action functional, the following positive determinant :

$$\Delta(w) := 1 - \frac{\sin^2 w + 4(1 - \cos w)}{w^2} + \frac{4(1 - \cos w) \sin w}{w^3},$$

and the following positive (i.e., non-negative non-degenerate) quadratic form $\psi(w, \cdot, \cdot)$:

$$\begin{aligned} \psi(w, y, z) = & \left(1 + \frac{\sin(2w)}{2w} - 2\left(\frac{\sin w}{w}\right)^2 \right) \left(\frac{\sin w}{w} - y \right)^2 + \left(1 - \frac{\sin(2w)}{2w} - 2\left(\frac{1 - \cos w}{w}\right)^2 \right) \left(\frac{1 - \cos w}{w} - z \right)^2 \\ & - 4 \left(\frac{\sin^2 w}{2w} - \frac{(1 - \cos w) \sin w}{w^2} \right) \left(\frac{\sin w}{w} - y \right) \left(\frac{1 - \cos w}{w} - z \right). \end{aligned}$$

(ii) First degenerate case. For any $y \leq 0$, we uniformly have :

$$p_\varepsilon(0; (0, y, 0)) = \exp \left[-4\pi^2 \frac{\varepsilon - y}{\varepsilon^2} \right] \times \frac{2\sqrt{2}e + o(1)}{\varepsilon^3 \sqrt{\varepsilon - y}} \times \int_0^\infty \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right)}{(x^2 + 1)^{1/4}} dx.$$

(iii) *Second degenerate case.* For any $(y, z) \in \mathbb{R}^2$ such that $z \neq 0$ or $y > 0$, we have :

$$p_\varepsilon(0; (0, y, z)) = \frac{\exp[-\pi^2 C_\varepsilon(y, z)]}{\varepsilon^4 C_\varepsilon(y, z)^{3/4}} \times \frac{(2\pi/\text{sh } \pi)^{1/4}}{\sqrt{\pi - 2 \text{th } \frac{\pi}{2}}} \int_0^\infty \frac{\sin(\frac{3\pi}{16} + x)}{x^{1/4}} dx (1 + o(1)),$$

$$\text{with } C_\varepsilon(y, z) := \frac{\pi z^2}{2(\pi - 2 \text{th } \frac{\pi}{2}) \varepsilon^3} + \frac{y - \varepsilon}{\varepsilon^2}.$$

Remark 2.2 The statement in Theorem 2.1 is written when starting from 0. Actually this covers the generic initial value as well, by homogeneity, owing to the obvious action of the underlying affine rotation group. Namely, we merely have :

$$\begin{aligned} & p_\varepsilon((w_0, y_0, z_0); (w, y, z)) \\ &= p_\varepsilon((0, 0, 0); (w - w_0, (y - y_0) \cos w_0 + (z - z_0) \sin w_0, (z - z_0) \cos w_0 - (y - y_0) \sin w_0)). \end{aligned}$$

Remark 2.3 The coefficients $\Delta(w)$ and $\psi(w, y, z)$ that appear in the non-degenerate case of the statement come essentially from some inverse Malliavin matrix $(DU^0)^{-1}$; in particular (see Remark 4.1.3 and Section 5 below), we have $\Delta(w) = 4w^4 \det(DU^0)$ ($= \frac{w^6}{2160} + \mathcal{O}(w^8)$) for small $|w|$, which lets appear the singularity of the case $w = 0$ and

$$\psi(w, y, z) = 2w^2 \left(y - \frac{\sin w}{w}, \frac{1 - \cos w}{w} - z \right) \times DU^0 \times^t \left(y - \frac{\sin w}{w}, \frac{1 - \cos w}{w} - z \right),$$

which vanishes if and only if $\frac{\sin w}{w} - y = \frac{1 - \cos w}{w} - z = 0$.

In polar coordinates $y = \varrho \cos \alpha$, $z = \varrho \sin \alpha$, this reads

$$\begin{aligned} \psi(w, y, z) &= \varrho^2 \left[1 - \frac{\sin w}{w} \cos(w - 2\alpha) - \frac{2(1 - \cos w)}{w^2} (1 - \cos(w - 2\alpha)) \right] \\ &\quad - 2\varrho \left[1 - \frac{\sin w}{w} \right] \frac{\sin(w - \alpha) + \sin \alpha}{w} + 2 \left[1 - \frac{\sin w}{w} \right] \frac{1 - \cos w}{w^2}. \end{aligned}$$

Remark 2.4 The expressions given by Theorem 2.1 are only asymptotically valid. Indeed, starting from 0, by the very definition of (x_ε) for any positive time ε we have $y_\varepsilon^2 + z_\varepsilon^2 \leq \varepsilon^2$, so that $p_\varepsilon(0; (w, y, z))$ must vanish for $y^2 + z^2 \geq \varepsilon^2$. This is coherent with Theorem 2.1 (as $\lim_{\varepsilon \rightarrow 0} p_\varepsilon(0; \cdot) = 0$), but forbids to think of these asymptotics as more widely valid.

Remark 2.5 The initial null speed of the third component z_s of x_s explains the stronger energy in the exponent (iii) when $z \neq 0$. Similarly, the initial positive speed of the second component y_s makes more difficult that it rapidly reaches a non-positive value y , than a positive one. This explains the above difference between the exponents of p_s , regarding both cases (ii) and [(iii), $z = 0$]. Finally the presence of $(y - \varepsilon)$ in the exponents is natural, owing to the deterministic contribution at time ε (corresponding to w remaining at 0).

Remark 2.6 Though there is no natural underlying metric in this strictly hypoelliptic framework, the exponents given by Theorem 2.1 describe some action-energy functionals, and furthermore the singular case $w = 0$ lets think of a cut-locus which would be relative to some absent or hidden metric.

Moreover, these exponents show important differences with respect to the sub-elliptic case, generically handled in [BA1] for the non-degenerate case, and then partially (precisely, on the diagonal, which corresponds to $w = y = z = 0$ here) in [BA2] for the degenerate case. Indeed, on the one hand the energy functional in Theorem 2.1(i) above cannot be expressed as $d^2(0, x)/2\varepsilon$ (as in the sub-elliptic case) and even hardly as $d_\varepsilon^2(0, x)/2\varepsilon$ (as in the flat Langevin case), and on the other hand the decay factor is polynomial in the diagonal sub-elliptic case, while exponential in Theorem 2.1(ii)($y = 0$) above.

3 Pinned Fourier transform

Recall that p_ε denotes the density of $\left(\sqrt{\varepsilon}\omega_1, \varepsilon \int_0^1 \cos(\sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(\sqrt{\varepsilon}\omega_\tau) d\tau\right)$.

3.1 Brownian bridge disintegration

Let us condition on $\omega_1 = \frac{w}{\sqrt{\varepsilon}}$, and use the Brownian bridge $(\omega_s, 0 \leq s \leq 1)$ from 0 to $\frac{w}{\sqrt{\varepsilon}}$, whose law will be denoted by $\mathbb{P}_0^{w/\sqrt{\varepsilon}}$, to first disintegrate the Brownian law.

Lemma 3.1.1 *For each real w , $\sqrt{2\pi\varepsilon} e^{w^2/2\varepsilon} p_\varepsilon(0; (w, \cdot, \cdot))$ is the density of $\left(\varepsilon \int_0^1 \cos(w\tau + \sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(w\tau + \sqrt{\varepsilon}\omega_\tau) d\tau\right)$ under the Brownian bridge law \mathbb{P}_0^0 .*

Proof For all test functions φ, Φ , denoting by \mathbb{P}_0 the standard Brownian law we have:

$$\begin{aligned} & \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} \Phi(u, v) p_\varepsilon(0; (w, u, v)) du dv \right] \varphi(w) dw \\ &= \mathbb{E}_0 \left[\varphi\left(\sqrt{\varepsilon}\omega_1\right) \Phi\left(\varepsilon \int_0^1 \cos(\sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(\sqrt{\varepsilon}\omega_\tau) d\tau\right) \right] \\ &= \int_{\mathbb{R}} e^{-w^2/2\varepsilon} \mathbb{E}_0^{w/\sqrt{\varepsilon}} \left[\varphi\left(\sqrt{\varepsilon}\omega_1\right) \Phi\left(\varepsilon \int_0^1 \cos(\sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(\sqrt{\varepsilon}\omega_\tau) d\tau\right) \right] \frac{dw}{\sqrt{2\pi\varepsilon}} \\ &= \int_{\mathbb{R}} \frac{e^{-w^2/2\varepsilon}}{\sqrt{2\pi\varepsilon}} \mathbb{E}_0^0 \left[\Phi\left(\varepsilon \int_0^1 \cos(\tau w + \sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(\tau w + \sqrt{\varepsilon}\omega_\tau) d\tau\right) \right] \varphi(w) dw, \end{aligned}$$

so that

$$\begin{aligned} & \int_{\mathbb{R}^2} \Phi(u, v) p_\varepsilon(0; (w, u, v)) du dv \\ &= \frac{e^{-w^2/2\varepsilon}}{\sqrt{2\pi\varepsilon}} \mathbb{E}_0^0 \left[\Phi\left(\varepsilon \int_0^1 \cos(\tau w + \sqrt{\varepsilon}\omega_\tau) d\tau, \varepsilon \int_0^1 \sin(\tau w + \sqrt{\varepsilon}\omega_\tau) d\tau\right) \right]. \quad \diamond \end{aligned}$$

3.2 Fourier-Plancherel Formula

We here use the Fourier transform and the Plancherel inversion formula, as ([BA1], (3.9) p.319), but without potential function. Note that the smooth density $(w, u, v) \mapsto p_\varepsilon(0; (w, u, v))$ is bounded and integrable, hence square integrable. Then we dilate the integration variable according to $(\xi', \xi) \mapsto (\varepsilon^{-3/2} \xi', \varepsilon^{-3/2} \xi)$. Thus we deduce from Lemma 3.1.1 that we have :

$$\begin{aligned}
& 4\pi^2 \varepsilon^3 \sqrt{2\pi \varepsilon} \times e^{w^2/(2\varepsilon)} \times p_\varepsilon(0; (w, \varepsilon y, \varepsilon z)) \\
&= \varepsilon^3 \sqrt{2\pi \varepsilon} e^{w^2/(2\varepsilon)} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{\sqrt{-1}(\xi' u + \xi v)} p_\varepsilon(0; (w, u, v)) du dv \right] e^{-\sqrt{-1}(\xi' \varepsilon y + \xi \varepsilon z)} d\xi' d\xi \\
&= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{\sqrt{-1} \left(\frac{\xi' u}{\varepsilon^{3/2}} + \frac{\xi v}{\varepsilon^{3/2}} \right)} \sqrt{2\pi \varepsilon} e^{\frac{w^2}{2\varepsilon}} p_\varepsilon(0; (w, u, v)) dudv \right] e^{-\sqrt{-1} \left(\frac{\xi' y}{\varepsilon^{1/2}} + \frac{\xi z}{\varepsilon^{1/2}} \right)} d\xi' d\xi \\
&= \int_{\mathbb{R}^2} \mathbb{E}_0^0 \left[\exp \left(\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\int_0^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds - y \right] + \xi \left[\int_0^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds - z \right] \right) \right) \right] d\xi' d\xi.
\end{aligned}$$

We re-state this as follows.

Lemma 3.2.1 *For any $\varepsilon > 0$ and $(w, y, z) \in \mathbb{R}^3$, we have :*

$$p_\varepsilon(0; (w, \varepsilon y, \varepsilon z)) = \frac{e^{-w^2/(2\varepsilon)}}{4\pi^2 \varepsilon^3 \sqrt{2\pi \varepsilon}} \int_{\mathbb{R}^2} P_\varepsilon(\xi', \xi) d\xi' d\xi, \quad (3)$$

with

$$P_\varepsilon(\xi', \xi) := \mathbb{E}_0^0 \left[\exp \left(\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\int_0^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds - y \right] + \xi \left[\int_0^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds - z \right] \right) \right) \right].$$

This lemma incites to compare $P_\varepsilon(\xi', \xi)$ with

$$\begin{aligned}
\overline{P}_\varepsilon(\xi', \xi) &:= \exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\int_0^1 \cos(ws) ds - y \right] + \xi \left[\int_0^1 \sin(ws) ds - z \right] \right) \right] \times \\
&\quad \times \mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 (\xi \cos(ws) - \xi' \sin(ws)) \omega_s ds \right) \right].
\end{aligned}$$

However this comparison needs careful estimates, which must overcome integration against $d\xi' d\xi$, and constitute the content of the following section 4. Though the replacement of $P_\varepsilon(\xi', \xi)$ by $\overline{P}_\varepsilon(\xi', \xi)$ seems a priori reasonable, we have indeed to be very careful, owing to the high sensibility of oscillatory integrals : see for example Section 4 of [I-M].

4 Analysis and domination of $P_\varepsilon(\xi', \xi)$

By Lemma 3.2.1 and the order 1 Taylor formula we have

$$P_\varepsilon(\xi', \xi) = \exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\int_0^1 \cos(ws) ds - y \right] + \xi \left[\int_0^1 \sin(ws) ds - z \right] \right) \right] \times \mathcal{E}_\varepsilon(\xi', \xi), \quad (4)$$

with

$$\mathcal{E}_\varepsilon(\xi', \xi) := \mathbb{E}_0^0 \left[\exp(\sqrt{-1} A_\varepsilon(\xi', \xi)) \right] \quad (5)$$

and

$$A_\varepsilon(\xi', \xi) := \int_0^1 \int_0^1 [\xi \cos(ws + \sqrt{\varepsilon} \tau \omega_s) - \xi' \sin(ws + \sqrt{\varepsilon} \tau \omega_s)] \omega_s ds d\tau. \quad (6)$$

Let us decompose the phase according to $A_\varepsilon(\xi', \xi) = \xi' U_1^\varepsilon + \xi U_2^\varepsilon$, where for $\varepsilon \geq 0$:

$$U^\varepsilon(\omega) := \left(- \int_{[0,1]^2} \sin(ws + \tau \sqrt{\varepsilon} \omega_s) d\tau \omega_s ds, \int_{[0,1]^2} \cos(ws + \tau \sqrt{\varepsilon} \omega_s) d\tau \omega_s ds \right). \quad (7)$$

Thus we have to analyse the behaviour of $\mathcal{E}_\varepsilon(\xi', \xi)$ as $\varepsilon \searrow 0$. As the term $A_\varepsilon(\xi', \xi)$ is clearly almost surely continuous with respect to ε , this amounts to dominating $\mathcal{E}_\varepsilon(\xi', \xi)$ with respect to the Lebesgue measure $d\xi' d\xi$ on \mathbb{R}^2 .

To proceed, we shall use Theorem 2.1 of [T2], viewing $\mathcal{E}_\varepsilon(\xi', \xi)$ as an infinite-dimensional oscillatory integral with phase $A_\varepsilon(\xi', \xi)$. This requires a careful analysis of the (non-deterministic) Malliavin matrix DU^ε of U^ε .

4.1 Analysis of the phase vector U^ε (defined by (7))

In order to control the error term by a $d\xi' d\xi$ -integrable asymptotically vanishing term, we resort to Malliavin's calculus, as in [BA1], to take advantage of the oscillatory nature of the integrand, which has phase $A_\varepsilon(\xi', \xi)$. Namely we shall perform the analogue of Lemma (3.48) in [BA1], in order to obtain a $d\xi' d\xi$ -integrable control on $P_\varepsilon(\xi', \xi)$. However the non-deterministic nature of the Malliavin matrix DU^ε we have to face here causes difficulties that did not arise in [BA1]. We shall thus use Malliavin's Calculus in another way than [BA1], namely with the help of [T2] in order to fully take advantage of the oscillatory nature of the error term (which crude L^p estimates would ruin).

4.1.1 Integration by parts with respect to \mathbb{P}_0^0

Let us use integration by parts with respect to the pinned Wiener measure \mathbb{P}_0^0 .

For an accessible reference on Malliavin's Calculus, see for example the book [Fa] or [M].

Classically denoting by \mathcal{W}_0 the pinned Wiener space of continuous real maps ω on $[0, 1]$ such that $\omega_0 = \omega_1 = 0$ and by H_0 the associated Cameron-Martin space, we successively have (F, G denoting real-valued elements of $D_1^2(\mathcal{W}_0)$, and Z any element of $C(\mathcal{W}_0, H_0)$):

the gradient ∇ , defined for $h \in H_0, \omega \in \mathcal{W}_0$ by $\langle \nabla F(\omega), h \rangle_{H_0} \equiv \nabla_h F(\omega) := \frac{d_o}{dt} F(\omega + th)$;

the divergence operator or Skorohod integral δ , given by $\delta(Z)(\omega) := \int_0^1 \dot{Z}_s(\omega) D\omega_s$, and in

particular: $\delta(Fh)(\omega) = F(\omega) \int_0^1 \dot{h}(s) d\omega_s - \nabla_h F$, $\delta\left(\int_0^1 f(t, \omega_t) dt\right) = \int_0^1 f(t, \omega_t) d\omega_t$;

the integration by parts formula : $\mathbb{E}_0^0[\langle \nabla F, \nabla G \rangle_{H_0}] = \mathbb{E}_0^0[\delta(\nabla F) \times G]$;

the Ornstein-Uhlenbeck operator \mathcal{L} , given by $\mathcal{L}F := \delta(\nabla F)$; the energy estimate (see for example [Fa] page 57) $\mathbb{E}_0^0[(\mathcal{L}F)^2] \leq 4 \mathbb{E}_0^0[\|\nabla^2 F\|_{H_0 \otimes H_0}^2 + \|\nabla F\|_{H_0}^2]$.

4.1.2 Gradients and Malliavin matrix of U^ε

According to (7), for any $h \in H_0$ we have :

$$\begin{aligned} \langle \nabla U_2^\varepsilon(\omega), h \rangle_{H_0} &= \nabla_h U_2^\varepsilon(\omega) = \int_{[0,1]^2} [\cos(ws + \sqrt{\varepsilon} \tau \omega_s) - \sqrt{\varepsilon} \tau \omega_s \sin(ws + \sqrt{\varepsilon} \tau \omega_s)] h_s ds d\tau \\ &= \int_0^1 \cos(ws + \sqrt{\varepsilon} \omega_s) h_s ds = \int_0^1 \left[\int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds \right] \dot{h}_t dt, \end{aligned}$$

i.e.,

$$\frac{d}{dt} \nabla U_2^\varepsilon(\omega)(t) = \int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds - \int_0^1 du \int_u^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds = \frac{d}{dt} \nabla U_2^1(\sqrt{\varepsilon} \omega)(t) ;$$

similarly,

$$\frac{d}{dt} \nabla U_1^\varepsilon(\omega)(t) = \int_0^1 du \int_u^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds - \int_t^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds = \frac{d}{dt} \nabla U_1^1(\sqrt{\varepsilon} \omega)(t) ;$$

and

$$\frac{d}{dt} \nabla U^0(\omega)(t) = \left(\frac{\sin w}{w^2} - \frac{\cos(wt)}{w}, \frac{1 - \cos w}{w^2} - \frac{\sin(wt)}{w} \right) = \frac{d}{dt} \nabla U^0(0)(t) = \frac{d}{dt} \nabla U^\varepsilon(0)(t) .$$

Then the Malliavin covariance matrix is :

$$DU^\varepsilon(\omega) := \begin{pmatrix} \langle \nabla U_1^\varepsilon, \nabla U_1^\varepsilon \rangle_{H_0} & \langle \nabla U_1^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0} \\ \langle \nabla U_2^\varepsilon, \nabla U_1^\varepsilon \rangle_{H_0} & \langle \nabla U_2^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0} \end{pmatrix}(\omega) = DU^1(\sqrt{\varepsilon} \omega), \quad (8)$$

with

$$\langle \nabla U_1^\varepsilon, \nabla U_1^\varepsilon \rangle_{H_0} = \int_0^1 \left(\int_t^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds \right)^2 dt - \left[\int_0^1 \left(\int_t^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds \right) dt \right]^2 \leq \frac{1}{3},$$

$$\langle \nabla U_2^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0} = \int_0^1 \left(\int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds \right)^2 dt - \left[\int_0^1 \left(\int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds \right) dt \right]^2 \leq \frac{1}{3},$$

so that $\text{tr}(DU^\varepsilon) \geq 2\sqrt{\det(DU^\varepsilon)} \geq 6 \det(DU^\varepsilon)$, and

$$\begin{aligned} \langle \nabla U_1^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0} &= \int_0^1 \left[\int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds \right] dt \times \int_0^1 \left[\int_t^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds \right] dt \\ &\quad - \int_0^1 \left[\int_t^1 \cos(ws + \sqrt{\varepsilon} \omega_s) ds \right] \times \left[\int_t^1 \sin(ws + \sqrt{\varepsilon} \omega_s) ds \right] dt. \end{aligned}$$

Remark 4.1.3 In particular, at $\varepsilon = 0$ we have the following deterministic limiting Malliavin covariance matrix :

$$DU^0 := \begin{pmatrix} \frac{1}{2w^2} \left(1 + \frac{\sin(2w)}{2w} \right) - \frac{\sin^2 w}{w^4} & \frac{\sin^2 w}{2w^3} - \frac{(1-\cos w) \sin w}{w^4} \\ \frac{\sin^2 w}{2w^3} - \frac{(1-\cos w) \sin w}{w^4} & \frac{1}{2w^2} \left(1 - \frac{\sin(2w)}{2w} \right) - \frac{(1-\cos w)^2}{w^4} \end{pmatrix},$$

and then

$$\begin{aligned} \det(DU^0) &= \det(DU^0)(0) = \det(DU^\varepsilon)(0) \\ &= \left[\frac{1}{2w^2} \left(1 - \frac{\sin(2w)}{2w} \right) - \frac{(1-\cos w)^2}{w^4} \right] \times \left[\frac{1}{2w^2} \left(1 + \frac{\sin(2w)}{2w} \right) - \frac{\sin^2 w}{w^4} \right] - \left[\frac{\sin^2 w}{2w^3} - \frac{(1-\cos w) \sin w}{w^4} \right]^2 \\ &= \frac{1}{4w^4} - \frac{4(1-\cos w) + \sin^2 w}{4w^6} + \frac{(1-\cos w) \sin w}{w^7} = \frac{w^2}{8640} + \mathcal{O}(w^4) \quad (\text{for small } |w|). \end{aligned}$$

This determinant $\det(DU^0)$ would vanish if and only if both centred functions $\frac{\sin w}{w^2} - \frac{\cos(wt)}{w}$ and $\frac{1-\cos w}{w^2} - \frac{\sin(wt)}{w}$ were proportional; but this would imply proportionality between $t \mapsto \cos(wt)$ and $t \mapsto \sin(wt)$, which does not hold provided $w \neq 0$. Hence $\det(DU^0) > 0$ for any $w \neq 0$. For the same reason, and since \mathbb{P}_0^0 -almost surely $\omega \neq [s \mapsto -ws/\sqrt{\varepsilon}]$, we almost surely have $\det(DU^\varepsilon) > 0$, for any $\varepsilon \geq 0$.

Then we have

$$\mathcal{L}U_2^\varepsilon(\omega) = \int_0^1 \frac{d}{dt} \nabla U_2^\varepsilon(\omega)(t) d\omega_t = \int_0^1 \cos(ws + \sqrt{\varepsilon} \omega_s) \omega_s ds, \quad (9)$$

$$\text{and similarly } \mathcal{L}U_1^\varepsilon(\omega) = - \int_0^1 \sin(ws + \sqrt{\varepsilon} \omega_s) \omega_s ds.$$

Thence

$$\begin{aligned} \frac{d}{dt} \nabla \mathcal{L}U_1^\varepsilon(\omega)(t) &= \int_0^t [\sqrt{\varepsilon} \omega_s \cos(ws + \sqrt{\varepsilon} \omega_s) + \sin(ws + \sqrt{\varepsilon} \omega_s)] ds - \int_0^1 \int_0^t \text{idem}; \\ \frac{d}{dt} \nabla \mathcal{L}U_2^\varepsilon(\omega)(t) &= \int_0^t [\sqrt{\varepsilon} \omega_s \sin(ws + \sqrt{\varepsilon} \omega_s) - \cos(ws + \sqrt{\varepsilon} \omega_s)] ds - \int_0^1 \int_0^t \text{idem}; \\ \|\nabla \mathcal{L}U_1^\varepsilon\|_{H_0}^2(\omega) &= \text{Var}_{[0,1]} \left[\int_0^\bullet [\sqrt{\varepsilon} \omega_s \cos(ws + \sqrt{\varepsilon} \omega_s) + \sin(ws + \sqrt{\varepsilon} \omega_s)] ds \right] \leq 1 + \varepsilon \int_0^1 \omega^2; \\ \|\nabla \mathcal{L}U_2^\varepsilon\|_{H_0}^2(\omega) &= \text{Var}_{[0,1]} \left[\int_0^\bullet [\sqrt{\varepsilon} \omega_s \sin(ws + \sqrt{\varepsilon} \omega_s) - \cos(ws + \sqrt{\varepsilon} \omega_s)] ds \right] \leq 1 + \varepsilon \int_0^1 \omega^2. \end{aligned}$$

4.1.4 Estimates for the successive gradients of U^ε and $\mathcal{L}U^\varepsilon$

We begin with an explicit formula for $\nabla^k U_2^\varepsilon$.

Lemma 4.1.5 For $k \geq 1$, setting

$$g_k^\omega(t) := (\sqrt{\varepsilon})^{k-1} \int_t^1 \cos^{(k-1)}(wu + \sqrt{\varepsilon} \omega_u) du,$$

we almost surely have (recall the definition (7) of U^ε):

$$\frac{d^k}{ds_1 \dots ds_k} \nabla^k U_2^\varepsilon(\omega)(s_1, \dots, s_k) = \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} g_k^\omega(\max\{s_1, \dots, s_k\}) ds_{i_1} \dots ds_{i_j}.$$

Proof We proceed by induction on k . We already saw the case $k = 1$ in the beginning of Section 4.1.2. Then set $\delta_k := \frac{d^k}{ds_1 \dots ds_k} \nabla^k U_2^\varepsilon(\omega)$, denote by Δ_k the right hand side of the above formula, and assume that $\delta_k(s_1, \dots, s_k) = \Delta_k$. Then for any $h \in H_0^1$ and any $t \in [0, 1]$ we have

$$\begin{aligned} \nabla_h g_k^\omega(t) &= \varepsilon^{k/2} \int_t^1 \cos^{(k)}(wu + \sqrt{\varepsilon} \omega_u) h(u) du \\ &= \varepsilon^{k/2} \int_0^1 \int_s^1 1_{[t,1]}(u) \cos^{(k)}(wu + \sqrt{\varepsilon} \omega_u) du \dot{h}(s) ds = \int_0^1 g_{k+1}^\omega(\max\{s, t\}) \dot{h}(s) ds. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^1 \delta_{k+1}(s_1, \dots, s_{k+1}) \dot{h}(s_{k+1}) ds_{k+1} &= \nabla_h [\delta_k(s_1, \dots, s_k)] = \nabla_h \Delta_k \\ &= \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} \nabla_h g_k^\omega(\max\{s_1, \dots, s_k\}) ds_{i_1} \dots ds_{i_j} \\ &= \int_0^1 \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} g_{k+1}^\omega(\max\{s_1, \dots, s_{k+1}\}) ds_{i_1} \dots ds_{i_j} \dot{h}(s_{k+1}) ds_{k+1}, \end{aligned}$$

so that

$$\begin{aligned} \delta_{k+1}(s_1, \dots, s_{k+1}) &= \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} g_{k+1}^\omega(\max\{s_1, \dots, s_{k+1}\}) ds_{i_1} \dots ds_{i_j} \\ &\quad - \int_0^1 \left[\sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} g_{k+1}^\omega(\max\{s_1, \dots, s_{k+1}\}) ds_{i_1} \dots ds_{i_j} \right] ds_{k+1} \\ &= \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \left[\int_{[0,1]^j} g_{k+1}^\omega(\max\{s_1, \dots, s_{k+1}\}) ds_{i_1} \dots ds_{i_j} \right. \\ &\quad \left. - \int_{[0,1]^{j+1}} g_{k+1}^\omega(\max\{s_1, \dots, s_{k+1}\}) ds_{i_1} \dots ds_{i_j} ds_{k+1} \right] \\ &= \Delta_{k+1}. \quad \diamond \end{aligned}$$

Of course, the same holds with ‘sin’ instead of ‘cos’. This entails the following estimate.

Lemma 4.1.6 For $k \geq 1$ and $j = 0, 1$ we almost surely have :

$$\|\nabla^k U_j^\varepsilon\|_{H_0^{\otimes k}} \leq 2\sqrt{2} k^{-1} (2\sqrt{\varepsilon})^{k-1},$$

and (recall (6)) :

$$\|\nabla^k A_\varepsilon(\xi', \xi)\|_{H_0^{\otimes k}} \leq 4\sqrt{\xi^2 + \xi'^2} k^{-1} (2\sqrt{\varepsilon})^{k-1}.$$

Proof The latter follows at once from the former, owing to (6). Now by Lemma 4.1.6 for any $k \geq 1$ we have $|g_k^\omega(t)| \leq \varepsilon^{\frac{k-1}{2}} (1-t)$, and

$$\begin{aligned} \|\nabla^k U_j^\varepsilon\|_{H_0^{\otimes k}}^2 &= \int_{[0,1]^k} \delta_k(s_1, \dots, s_k)^2 ds_1 \dots ds_k = \int_{[0,1]^k} \Delta_k^2 ds_1 \dots ds_k \\ &\leq 4^k \int_{[0,1]^k} g_k^\omega(\max\{s_1, \dots, s_k\})^2 ds_1 \dots ds_k \leq 4^k \varepsilon^{k-1} \int_{[0,1]^k} (1 - \max\{s_1, \dots, s_k\})^2 ds_1 \dots ds_k \\ &= 4^k \varepsilon^{k-1} \times \frac{2}{(k+1)(k+2)} < 8 k^{-2} (4\varepsilon)^{k-1}. \quad \diamond \end{aligned}$$

We can proceed analogously to handle $\nabla^k \mathcal{L}U_2^\varepsilon$.

Lemma 4.1.7 For $k \geq 1$, setting

$$\tilde{g}_k^\omega(t) := (\sqrt{\varepsilon})^{k-1} \int_t^1 \left[k \cos^{(k-1)} + \sqrt{\varepsilon} \omega_u \cos^{(k)} \right] (wu + \sqrt{\varepsilon} \omega_u) du,$$

we almost surely have :

$$\frac{d^k}{ds_1 \dots ds_k} \nabla^k \mathcal{L}U_2^\varepsilon(\omega)(s_1, \dots, s_k) = \sum_{j=0}^k (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \int_{[0,1]^j} \tilde{g}_k^\omega(\max\{s_1, \dots, s_k\}) ds_{i_1} \dots ds_{i_j}.$$

Proof We proceed as for the above lemma 4.1.5. The case $k = 1$ reduces to (9) in Section 4.1.2. Then the induction on k is as in the proof of Lemma 4.1.5, except that the computation of $\nabla_h g_k^\omega(t)$ must be replaced by the following :

$$\begin{aligned} \nabla_h \tilde{g}_k^\omega(t) &= \varepsilon^{k/2} \int_t^1 \left[(k+1) \cos^{(k)} + \sqrt{\varepsilon} \omega_u \cos^{(k+1)} \right] (wu + \sqrt{\varepsilon} \omega_u) h(u) du \\ &= \varepsilon^{k/2} \int_0^1 \int_s^1 1_{[t,1]}(u) \left[(k+1) \cos^{(k)} + \sqrt{\varepsilon} \omega_u \cos^{(k+1)} \right] (wu + \sqrt{\varepsilon} \omega_u) du \dot{h}(s) ds \\ &= \int_0^1 \tilde{g}_{k+1}^\omega(\max\{s, t\}) \dot{h}(s) ds. \quad \diamond \end{aligned}$$

Of course, the same holds with ‘sin’ instead of ‘cos’. This entails the following estimate.

Lemma 4.1.8 For $k \geq 1$ and $j = 0, 1$ we almost surely have :

$$\|\nabla^k \mathcal{L}U_j^\varepsilon\|_{H_0^{\otimes k}} \leq 2\sqrt{2} (2\sqrt{\varepsilon})^{k-1} \times \left[2 + \varepsilon \int_0^1 \omega^2 \right]^{1/2}.$$

Proof More or less as for Lemma 4.1.6, but owing now to Lemma 4.1.7, on the one we have

$$|\tilde{g}_k^\omega(t)| \leq (\sqrt{\varepsilon})^{k-1} \int_t^1 \sqrt{k^2 + \varepsilon \omega_u^2} du \leq (\sqrt{\varepsilon})^{k-1} k(1-t) + (\sqrt{\varepsilon})^k \int_0^1 |\omega_u| du,$$

and on the other hand,

$$\begin{aligned} \|\nabla^k \mathcal{L}U_j^\varepsilon\|_{H_0^{\otimes k}}^2 &\leq 4^k \int_{[0,1]^k} \tilde{g}_k^\omega(\max\{s_1, \dots, s_k\})^2 ds_1 \dots ds_k \\ &\leq 4^k \varepsilon^{k-1} \times 2 \int_{[0,1]^k} \left(k^2 (1 - \max\{s_1, \dots, s_k\})^2 + \varepsilon \int_0^1 \omega^2 \right) ds_1 \dots ds_k \\ &< (4\varepsilon)^{k-1} \times 8 \left(2 + \varepsilon \int_0^1 \omega^2 \right). \diamond \end{aligned}$$

As a consequence, for the random variables

$$M_1[\varphi, r] := \sum_{n \geq 1} \frac{r^n}{n!} \|\nabla^n \varphi\|_{H_0^{\otimes n}}^2 \quad \text{and} \quad M_2[\varphi, r] := \sum_{n \geq 2} \frac{r^n}{n!} \|\nabla^n \varphi\|_{H_0^{\otimes n}}^2 \quad (10)$$

which will intervene in the following section 4.2, we directly deduce the following.

Lemma 4.1.9 *For any positive r, ε we almost surely have :*

$$M_1[U_1^\varepsilon, r] + M_1[U_2^\varepsilon, r] < 16 r e^{4r\varepsilon} \quad \text{and} \quad M_2[U_1^\varepsilon, r] + M_2[U_2^\varepsilon, r] < 16 r^2 \varepsilon (e^{4r\varepsilon} - 1);$$

$$M_1[\mathcal{L}U_1^\varepsilon, r] + M_1[\mathcal{L}U_2^\varepsilon, r] < 16 r e^{4r\varepsilon} \left[2 + \varepsilon \int_0^1 \omega^2 \right];$$

$$M_2[\mathcal{L}U_1^\varepsilon, r] + M_2[\mathcal{L}U_2^\varepsilon, r] < 16 r^2 \varepsilon (e^{4r\varepsilon} - 1) \left[2 + \varepsilon \int_0^1 \omega^2 \right].$$

4.2 Applying Theorem 2.1 of [T2]

We use the treatment of infinite-dimensional oscillatory integrals as developed in [T2], to estimate $\mathcal{E}_s(\xi', \xi)$, recall (5), (6). The correspondence between the notation used by Taniguchi S. in [T2] and ours is given by :

$$N = 1, \psi = 1, q_0 = U_1^\varepsilon, q_1 = U_2^\varepsilon, \lambda = \sqrt{\xi^2 + \xi'^2}, z^0 = \frac{\xi}{\lambda}, z^1 = \frac{\xi'}{\lambda}, A_\varepsilon(\xi', \xi) = \lambda q^{(z)}.$$

Thus $\|\nabla q^{(z)}\|_{H_0}^2 = \|z^1 \nabla U_1^\varepsilon + z^0 \nabla U_2^\varepsilon\|_{H_0}^2 = (z^1, z^0)(DU^\varepsilon)^t(z^1, z^0)$, so that

$$\begin{aligned} \min \left\{ \|\nabla q^{(z)}\|_{H_0}^2 \mid |z| = 1 \right\} &= \frac{\|\nabla U_1^\varepsilon\|_{H_0}^2 + \|\nabla U_2^\varepsilon\|_{H_0}^2 - \sqrt{(\|\nabla U_1^\varepsilon\|_{H_0}^2 - \|\nabla U_2^\varepsilon\|_{H_0}^2)^2 + 4 \langle \nabla U_1^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0}^2}}{2} \\ &= \frac{2 \det(DU^\varepsilon)}{\|\nabla U_1^\varepsilon\|_{H_0}^2 + \|\nabla U_2^\varepsilon\|_{H_0}^2 + \sqrt{(\|\nabla U_1^\varepsilon\|_{H_0}^2 - \|\nabla U_2^\varepsilon\|_{H_0}^2)^2 + 4 \langle \nabla U_1^\varepsilon, \nabla U_2^\varepsilon \rangle_{H_0}^2}} \geq \frac{\det(DU^\varepsilon)}{\text{tr}(DU^\varepsilon)} \geq \det(DU^\varepsilon). \quad (11) \end{aligned}$$

Then the estimate (2.1) in Theorem 2.1 of [T2] provides the following :

There exists $C > 0$ such that for any $r \in]1, e[$ and $\xi^2 + \xi'^2 > 1$, we have

$$|\mathcal{E}_\varepsilon(\xi', \xi)|^2 \leq C \mathbb{E}_0^0 \left[\exp \left(\frac{L_2[U^\varepsilon, r]}{L_1[U^\varepsilon, 1]^3} \right) \right] \times \mathbb{E}_0^0 \left[\exp \left(- \frac{\min\{\frac{\det(DU^\varepsilon)}{\text{tr}(DU^\varepsilon)}, 1\}}{147 C L_1[U^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right]$$

hence according to Section 4.1.2 (recall that $\text{tr}(DU^\varepsilon) \geq 2\sqrt{\det(DU^\varepsilon)} \geq 6 \det(DU^\varepsilon)$) :

$$|\mathcal{E}_\varepsilon(\xi', \xi)|^2 \leq C \mathbb{E}_0^0 \left[\exp \left(\frac{L_2[U^\varepsilon, r]}{L_1[U^\varepsilon, 1]^3} \right) \right] \times \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(DU^\varepsilon)/\text{tr}(DU^\varepsilon)}{147 C L_1[U^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right], \quad (12)$$

where (for $i = 1, 2$)

$$L_i[U^\varepsilon, r] := \sum_{j=1}^2 (M_i[U_j^\varepsilon, r] + M_i[\mathcal{L}U_j^\varepsilon, r]) + \left[1 + \sum_{j=1}^2 (\mathcal{L}U_j^\varepsilon)^2 \right]^{1/2},$$

with M_1, M_2 given by (10).

Now by the above lemma 4.1.9 and (9), for $r \in [1, 2]$ and $\varepsilon < 1$ we have :

$$\begin{aligned} 1 &\leq L_2[U^\varepsilon, r] \leq L_1[U^\varepsilon, r] \leq 16 r e^{4r\varepsilon} \left[3 + \varepsilon \int_0^1 \omega^2 \right] + \left[1 + 2 \int_0^1 \omega^2 \right]^{1/2} \\ &\leq \mathcal{O}(1) + \left[2 \int_0^1 \omega^2 \right]^{1/2} + \mathcal{O}(\varepsilon) \int_0^1 \omega^2 \leq \mathcal{O}(1) + \omega^* + \mathcal{O}(\varepsilon)(\omega^*)^2 =: \Omega_\varepsilon(\omega), \end{aligned} \quad (13)$$

where $\omega^* := \max_{[0,1]} |\omega|$.

Now (see for example ([R-Y], I.(3.10)) or [B-O]) for any $t > 0$ we have

$$\mathbb{P}_0^0[\omega^* > \sqrt{t}] = \mathbb{P}_0 \left[\sup_{s \geq 0} \{|\beta_s| - s\} > t \right] < 2 e^{-2t}. \quad (14)$$

As a consequence, for some large enough constant C we have :

$$\begin{aligned} &\mathbb{E}_0^0 \left[\exp \left(\frac{L_2[U^\varepsilon, r]}{L_1[U^\varepsilon, 1]^3} \right) \right] \leq \mathbb{E}_0^0 \left[e^{\log \sqrt{C} + \omega^* + \varepsilon \log C (\omega^*)^2} \right] \\ &= \sqrt{C} \int_0^\infty \mathbb{P}_0^0 \left[\omega^* + \varepsilon \log C (\omega^*)^2 > \log u \right] du = \sqrt{C} \int_{\mathbb{R}} \mathbb{P}_0^0 \left[\omega^* > \frac{\sqrt{1 + 4s\varepsilon \log C} - 1}{2\varepsilon \log C} \right] e^s ds \\ &< \sqrt{C} + 2\sqrt{C} \int_0^\infty e^{s - \frac{(\sqrt{1 + 4s\varepsilon \log C} - 1)^2}{2\varepsilon^2 \log^2 C}} ds = \sqrt{C} + 2\sqrt{C} \int_0^\infty e^{\frac{\sqrt{1 + 4s\varepsilon \log C}}{\varepsilon^2 \log^2 C} - (\frac{2}{\varepsilon \log C} - 1)s} ds \\ &\leq C \quad \text{for small enough } \varepsilon. \end{aligned}$$

Then by (12), for large enough positive constant C and small enough positive ε we have :

$$|\mathcal{E}_\varepsilon(\xi', \xi)|^2 \leq C^2 \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(DU^\varepsilon)/\text{tr}(DU^\varepsilon)}{147 C L_1[U^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right]. \quad (15)$$

4.3 Domination of the decreasing integral of (15)

We here establish the key domination property for the ε -dependent decreasing integral arising from ([T2], Th.2.1(2.1)) and Section 4.2 above, in the non-singular case $w \neq 0$. Thus the aim of this section is to prove the following crucial estimate.

Proposition 4.3.1 *For any fixed positive C and $w \in \mathbb{R}^*$, we have*

$$\sup_{0 \leq \varepsilon \leq 1} \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(DU^\varepsilon)/\text{tr}(DU^\varepsilon)}{147 C L_1[U^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right] \in L^{1/2}(\mathbb{R}^2, d\xi' d\xi).$$

4.3.2 Beginning of the proof of Proposition 4.3.1

Set $\mathcal{R}_\varepsilon := \frac{\det(DU^\varepsilon)}{\text{tr}(DU^\varepsilon) L_1[U^\varepsilon, 1]}$ and denote by λ_ε the lowest eigenvalue of DU^ε , which is almost surely positive. By (13) we have $\mathcal{R}_\varepsilon \geq \frac{\det(DU^\varepsilon)}{\text{tr}(DU^\varepsilon) \Omega_\varepsilon} \geq \frac{\lambda_\varepsilon}{2 \Omega_\varepsilon}$.

Now according to Subsection 4.1.2, for any $v = (v_1, v_2) \in \mathbb{S}^1 \subset \mathbb{R}^2$ we have

$$\langle v, DU^\varepsilon v \rangle = \text{Var}_{[0,1]} \left[\int_{\bullet}^1 [v_1 \sin(ws + \sqrt{\varepsilon} \omega_s) - v_2 \cos(ws + \sqrt{\varepsilon} \omega_s)] ds \right],$$

so that

$$\lambda_\varepsilon = \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right].$$

Hence, for any deterministic positive R we have:

$$\mathbb{E}_0^0 [e^{-\mathcal{R}_\varepsilon \times R}] \leq \mathbb{E}_0^0 \left[\exp \left(- \frac{\lambda_\varepsilon R}{2 \Omega_\varepsilon} \right) \right] = \mathbb{E}_0^0 \left[e^{\frac{-R}{2 \Omega_\varepsilon} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} \right]. \quad (16)$$

Then, for some positive constant C and any $R > 3/2$, $0 \leq \varepsilon < 1$, we have:

$$\begin{aligned} \mathbb{E}_0^0 \left[e^{\frac{-R}{2 \Omega_\varepsilon} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} \right] &\leq \mathbb{E}_0^0 \left[e^{\frac{-R/C}{1 + \omega^* + \varepsilon(\omega^*)^2} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} \right] \\ &\leq \mathbb{P}_0^0 [\omega^* > R^{1/3}] + \mathbb{E}_0^0 \left[e^{\frac{-R/C}{1 + R^{1/3} + \varepsilon R^{2/3}} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} 1_{\{\omega^* \leq R^{1/3}\}} \right] \\ &\leq 2 e^{-2 R^{2/3}} + \mathbb{E}_0^0 \left[e^{\frac{-R^{1/3}}{2C} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} \right]. \end{aligned}$$

Note that we applied (14) to get the last line. Hence, according to (16), so far we obtain:

$$\sqrt{\mathbb{E}_0^0 [e^{-\mathcal{R}_\varepsilon \times R}]} \leq 2 e^{-R^{2/3}} + \sqrt{\mathbb{E}_0^0 \left[e^{-\frac{R^{1/3}}{2C} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]} \right]}. \quad (17)$$

4.3.3 Estimating the Variance from below

We must now estimate the last term in the above (17). We first write a tractable expression of the variance: setting

$$\phi_t^\theta := \int_t^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds, \quad (18)$$

we have $\text{Var}_{[0,1]}(\phi^\theta) = \frac{1}{2} \int_{[0,1]^2} (\phi_t^\theta - \phi_u^\theta)^2 dt du = \int_{0 \leq t < u \leq 1} (\phi_t^\theta - \phi_u^\theta)^2 dt du$, and then

$$\text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right] = \int_{0 \leq t < u \leq 1} \left[\int_t^u \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right]^2 dt du. \quad (19)$$

We shall strongly use this expression (19) of the variance, and the following remark.

Remark 4.3.4 For $0 \leq s < t \leq 1$, for any $p \geq 2$ and for some constant c_p we have $\mathbb{E}_0^0[|\omega_t - \omega_s|^{2p}] = \mathbb{E}_0[|B_t - B_s - (t-s)B_1|^{2p}] = \mathbb{E}[|\mathcal{N}(0; (t-s)(1-t+s))|^{2p}] \leq c_p |t-s|^p$.

According to ([R-Y], I.Theorem (2.1)), this entails that setting

$S_p(\omega) := \sup_{0 \leq t < u \leq 1} \frac{|\omega_t - \omega_u|}{|t-u|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}}} = S_p(-\omega)$ defines a random variable $S_p \in L^{2p}(\mathcal{W}_0, \mathbb{P}_0^0)$, such that

$$\sup_{t \leq s \leq u} |\omega_s - \omega_t| \leq S_p(\omega) \times |u-t|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}} \quad \text{for any } 0 \leq t < u \leq 1.$$

The following crucial estimate amounts in the present context to control the determinant of the Malliavin matrix from below, provided $w \neq 0$.

Proposition 4.3.5 For any non-null real $w \in \mathbb{R}^*$ and any $p \geq 2$, using the random variable $S_p \in L^{2p}(\mathcal{W}_0, \mathbb{P}_0^0)$ of Remark 4.3.4, we have:

$$\inf_{0 \leq \varepsilon \leq 1} \inf_{\theta} \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \sin(ws + \theta + \sqrt{\varepsilon} \omega_s) ds \right] \geq \frac{4}{5} \left[\frac{\min\{\pi^2, w^2\}}{2\pi^2} \right]^{\frac{10p-2-2/p}{p-1-1/p}} (|w| + S_p)^{\frac{-8p}{p-1-1/p}}.$$

Proof A delicate point here is to obtain the uniformity with respect to θ .

By periodicity and symmetry, we can of course restrict to $0 \leq \theta < \pi$.

Set $M_\varepsilon^+(\omega) := \max_{0 \leq s \leq 1} \{ws + \sqrt{\varepsilon} \omega_s\}$, $M_\varepsilon^-(\omega) := -\min_{0 \leq s \leq 1} \{ws + \sqrt{\varepsilon} \omega_s\}$ and $M_\varepsilon := M_\varepsilon^+ + M_\varepsilon^-$.

Note that $M_\varepsilon \geq |w|$ \mathbb{P}_0^0 -almost surely, for any real w .

Thus \mathbb{P}_0^0 -almost surely, the image of $[0, 1] \ni s \mapsto |\sin(ws + \theta + \sqrt{\varepsilon} \omega_s)|$ is the random interval $J_\varepsilon^\theta := |\sin([\theta - M_\varepsilon^-, \theta + M_\varepsilon^+])|$, image under $|\sin(\cdot)|$ of the interval $[\theta - M_\varepsilon^-, \theta + M_\varepsilon^+]$.

Now, on the one hand $\inf_{\theta} \max_{\theta} J_\varepsilon^\theta$ is attained for $\theta \in \frac{U_\varepsilon^- - U_\varepsilon^+}{2} + \pi\mathbb{Z}$ and equal to $\tilde{M}_\varepsilon := \sin\left(\frac{\min\{U_\varepsilon, \pi\}}{2}\right) \geq \min\{1, M_\varepsilon/\pi\} > 0$, and on the other hand, the minimal amplitude of

J_ε^θ is attained for $\theta \in \frac{U_\varepsilon^- - U_\varepsilon^+ + \pi}{2} + \pi\mathbb{Z}$ and equal to $\inf_{\theta} \text{range}(J_\varepsilon^\theta) = 2 \sin^2\left(\frac{\min\{U_\varepsilon, \pi\}}{4}\right) \geq \min\{1, M_\varepsilon^2/2\pi^2\} > 0$ (we let $\text{range}(J_\varepsilon^\theta) := \max J_\varepsilon^\theta - \min J_\varepsilon^\theta$).

Consider also $\widetilde{J}_\varepsilon^\theta := \max\{\min J_\varepsilon^\theta, \tilde{M}_\varepsilon/2\} \in J_\varepsilon^\theta$, and the random times :

(i) if $\widetilde{J}_\varepsilon^\theta(\omega)$ is hit before $\max J_\varepsilon^\theta(\omega)$:

$$\begin{aligned} T_1(\omega) &:= \min\left\{s \in [0, 1] \mid |\sin(ws + \theta + \sqrt{\varepsilon}\omega_s)| = \widetilde{J}_\varepsilon^\theta(\omega)\right\}; \\ T_2(\omega) &:= \min\left\{s \in [T_1(\omega), 1] \mid |\sin(ws + \theta + \sqrt{\varepsilon}\omega_s)| = \max J_\varepsilon^\theta(\omega)\right\}; \end{aligned}$$

(i) if not :

$$\begin{aligned} T_1(\omega) &:= \min\left\{s \in [0, 1] \mid |\sin(ws + \theta + \sqrt{\varepsilon}\omega_s)| = \max J_\varepsilon^\theta(\omega)\right\}; \\ T_2(\omega) &:= \min\left\{s \in [T_1(\omega), 1] \mid |\sin(ws + \theta + \sqrt{\varepsilon}\omega_s)| = \widetilde{J}_\varepsilon^\theta(\omega)\right\}. \end{aligned}$$

Then on the one hand we have $|\sin(ws + \theta + \sqrt{\varepsilon}\omega_s)| \geq \tilde{M}_\varepsilon/2$ for $T_1 \leq s \leq T_2$, and on the other hand, by Remark 4.3.4 we have :

$$\begin{aligned} \frac{1}{2} \min\{1, M_\varepsilon^2/\pi^2\} &\leq \min\{\tilde{M}_\varepsilon/2, \inf_{\theta} \text{range}(J_\varepsilon^\theta)\} \\ &\leq |\sin(w T_1(\omega) + \theta + \sqrt{\varepsilon}\omega_{T_1}) - \sin(w T_2(\omega) + \theta + \sqrt{\varepsilon}\omega_{T_2})| \\ &\leq |w| |T_2(\omega) - T_1(\omega)| + \sqrt{\varepsilon} S_p(\omega) \times |T_2(\omega) - T_1(\omega)|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}} \\ &\leq (|w| + \sqrt{\varepsilon} S_p(\omega)) \times |T_2(\omega) - T_1(\omega)|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}}, \end{aligned}$$

whence \mathbb{P}_0^0 -almost surely,

$$T_2(\omega) - T_1(\omega) \geq \left(\frac{\min\{1, M_\varepsilon^2/\pi^2\}/2}{|w| + \sqrt{\varepsilon} S_p(\omega)} \right)^{\left(\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}\right)^{-1}} = \left(\frac{\min\{1, M_\varepsilon^2/\pi^2\}/2}{|w| + \sqrt{\varepsilon} S_p(\omega)} \right)^{\frac{2p}{p-1-1/p}}.$$

Thence we deduce :

$$\begin{aligned} \int_{0 \leq t < u \leq 1} \left[\int_t^u \sin(ws + \theta + \sqrt{\varepsilon}\omega_s) ds \right]^2 dt du &\geq \int_{T_1 \leq t < u \leq T_2} \left| \int_t^u \sin(ws + \theta + \sqrt{\varepsilon}\omega_s) ds \right|^2 dt du \\ &\geq \int_{T_1 \leq t < u \leq T_2} [(u - t) \tilde{M}_\varepsilon/2]^2 dt du = (\tilde{M}_\varepsilon)^2 (T_2 - T_1)^4 / 48 \\ &\geq \frac{(\tilde{M}_\varepsilon)^2}{48} \left[\frac{\min\{\pi^2, M_\varepsilon^2\}}{2\pi^2} \right]^{\frac{8p}{p-1-1/p}} (|w| + \sqrt{\varepsilon} S_p)^{\frac{-8p}{p-1-1/p}} \\ &\geq \frac{4}{5} \left[\frac{\min\{\pi^2, M_\varepsilon^2\}}{2\pi^2} \right]^{\frac{10p-2-2/p}{p-1-1/p}} (|w| + \sqrt{\varepsilon} S_p)^{\frac{-8p}{p-1-1/p}}. \end{aligned}$$

By (19), the claim is now clear, using that $M_\varepsilon \geq |w| > 0$ \mathbb{P}_0^0 -almost surely. \diamond

4.3.6 End of the proof of Proposition 4.3.1

The above proposition 4.3.5 and (17) together provide: for any $w \in \mathbb{R}^*$,

$$\sqrt{\sup_{0 \leq \varepsilon \leq 1} \mathbb{E}_0^0[e^{-\mathcal{R}_\varepsilon \times R}]} \leq 2e^{-R^{2/3}} + \sqrt{\mathbb{E}_0^0 \left[\exp \left(-\frac{2R^{1/3}}{5C} \left[\frac{\min\{\pi^2, w^2\}}{2\pi^2} \right]^{\frac{10p-2-\frac{2}{p}}{p-1-\frac{1}{p}}} (|w| + S_p)^{\frac{-8p}{p-1-\frac{1}{p}}} \right) \right]}.$$

Now, for any positive random variable V and any deterministic $Y > 0$, $q \in \mathbb{N}^*$ we have:

$$\begin{aligned} \mathbb{E}[e^{-Y/V}] &= \int_0^1 \mathbb{P}[e^{-Y/V} > x] dx = \int_1^\infty \mathbb{P}[V > Y/\log x] x^{-2} dx = Y \int_0^\infty \mathbb{P}[tV > 1] e^{-Yt} dt \\ &\leq Y \mathbb{E}(V^q) \int_0^\infty t^q e^{-Yt} dt = q! \mathbb{E}(V^q) Y^{-q}. \end{aligned}$$

In particular we thus have

$$\mathbb{E}_0^0 \left[e^{-Y(|w|+S_p)^{\frac{-8p}{p-1-\frac{1}{p}}}} \right] \leq q! \mathbb{E}_0^0 \left[(|w| + S_p)^{\frac{8pq}{p-1-\frac{1}{p}}} \right] Y^{-q},$$

and then by the above, setting $C(p, q, w) := \left(\frac{5C}{2}\right)^{q/2} \left[\frac{2\pi^2}{\min\{\pi^2, w^2\}} \right]^{\frac{(5p-1-\frac{1}{p})q}{p-1-\frac{1}{p}}}$ we have:

$$\sqrt{\sup_{0 \leq \varepsilon \leq 1} \mathbb{E}_0^0[e^{-\mathcal{R}_\varepsilon \times R}]} \leq 2e^{-R^{2/3}} + C(p, q, w) \sqrt{q! \mathbb{E}_0^0 \left[(|w| + S_p)^{\frac{8pq}{p-1-\frac{1}{p}}} \right]} R^{-q/6}.$$

As a consequence, taking for example $q = 13$ (in order to have $1 - q/6 < -1$) and $p = 54$ (in order to have $\frac{8pq}{p-1-\frac{1}{p}} \leq 2p$; recall Remark 4.3.4: $S_p \in L^{2p}(\mathcal{W}_0, \mathbb{P}_0^0)$), we obtain:

$$\int_1^\infty \sqrt{\sup_{0 \leq \varepsilon \leq 1} \mathbb{E}_0^0[e^{-\mathcal{R}_\varepsilon \times R}]} R dR \leq 6 + 6C(54, 13, w) \sqrt{13! \mathbb{E}_0^0 \left[(|w| + S_p)^{2p} \right]} < \infty.$$

This completes the proof of Proposition 4.3.1. \diamond

As a consequence, we have the following key approximation result.

Proposition 4.3.7 *As $\varepsilon \searrow 0$, for any $w \in \mathbb{R}^*$ and uniformly with respect to $(y, z) \in \mathbb{R}^2$, we have*

$$\int_{\mathbb{R}^2} P_\varepsilon(\xi', \xi) d\xi' d\xi = (1 + o(1)) \int_{\mathbb{R}^2} \exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\frac{\sin w}{w} - y \right] + \xi \left[\frac{1 - \cos w}{w} - z \right] \right) \right] \mathcal{E}_0(\xi', \xi) d\xi' d\xi.$$

Proof By (4) we have

$$P_\varepsilon(\xi', \xi) = \exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\frac{\sin w}{w} - y \right] + \xi \left[\frac{1 - \cos w}{w} - z \right] \right) \right] \times \mathcal{E}_\varepsilon(\xi', \xi).$$

By (15) and Proposition 4.3.1 we have $\sup_{0 \leq \varepsilon \leq \varepsilon_0} |\mathcal{E}_\varepsilon(\xi', \xi)| \in L^1(\mathbb{R}^2, d\xi' d\xi)$, for some $\varepsilon_0 > 0$.

This provides the wanted domination of $P_\varepsilon(\xi', \xi)$, which allows to apply the Lebesgue theorem with respect to $d\xi' d\xi$, using the continuity (recall (5),(6)) of $\mathcal{E}_\varepsilon(\xi', \xi)$ at $\varepsilon = 0$. \diamond

5 Small-time equivalent for p_ε , provided $w \neq 0$

The above proposition 4.3.7 allows to substitute $\bar{P}_\varepsilon(\xi', \xi)$ for $P_\varepsilon(\xi', \xi)$ in Lemma 3.2.1. Doing this and owing to (5) and (6), we directly obtain the following.

Proposition 5.1 *For any $w \in \mathbb{R}^*$ and uniformly with respect to $(y, z) \in \mathbb{R}^2$, as $\varepsilon \searrow 0$ we have*

$$p_\varepsilon(0; (w, \varepsilon y, \varepsilon z)) = \frac{(1 + o(1)) e^{-w^2/2\varepsilon}}{4\pi^2 \varepsilon^3 \sqrt{2\pi \varepsilon}} \int_{\mathbb{R}^2} \bar{P}_\varepsilon(\xi', \xi) d\xi' d\xi,$$

with

$$\begin{aligned} \bar{P}_\varepsilon(\xi', \xi) = & \exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\frac{\sin w}{w} - y \right] + \xi \left[\frac{1 - \cos w}{w} - z \right] \right) \right] \times \\ & \times \mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 (\xi \cos(ws) - \xi' \sin(ws)) \omega_s ds \right) \right]. \end{aligned}$$

Now we have the following easy lemma, which merely amounts to a Gaussian computation, or can as well be seen as the definition of the Gaussian (pinned Wiener) measure \mathbb{P}_0^0 on the Wiener space (\mathcal{W}_0, H_0) of the standard Brownian bridge.

Lemma 5.2 *For any complex deterministic continuous function u on $[0, 1]$, we have*

$$\mathbb{E}_0^0 \left[e^{\int_0^1 u_s \omega_s ds} \right] = \mathbb{E}_0^0 \left[e^{\int_0^1 (\int_s^1 u_\cdot) d\omega_s} \right] = \exp \left(\frac{1}{2} \left[\int_0^1 \left(\int_s^1 u \right)^2 ds - \left(\int_0^1 \left(\int_s^1 u \right) ds \right)^2 \right] \right).$$

Therefore we have

$$\mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 (\xi \cos(ws) - \xi' \sin(ws)) \omega_s ds \right) \right] = \exp \left[-\frac{1}{2} (\xi, \xi') (DU^0)^t (\xi, \xi') \right].$$

Then setting $\lambda := \frac{1 - \cos w}{w} - z$ and $\mu := \frac{\sin w}{w} - y$, we have :

$$\begin{aligned} \int_{\mathbb{R}^2} \bar{P}_\varepsilon(\xi', \xi) d\xi' d\xi &= \int_{\mathbb{R}^2} \exp \left[-\frac{1}{2} (\xi, \xi') (DU^0)^t (\xi, \xi') + \frac{\sqrt{-1}}{\sqrt{\varepsilon}} (\lambda \xi + \mu \xi') \right] d\xi' d\xi \\ &= \frac{2\pi}{\sqrt{\det(U^0)}} \exp \left(\frac{-1}{2\varepsilon} (\lambda, \mu) (DU^0)^{-1} {}^t(\lambda, \mu) \right) = \frac{2\pi}{\sqrt{\det(U^0)}} \exp \left(-\frac{\psi(w, y, z)}{4w^2 \varepsilon \det(U^0)} \right), \end{aligned}$$

with

$$\psi(w, y, z) := 2w^2 \left(y - \frac{\sin w}{w}, \frac{1 - \cos w}{w} - z \right) \times DU^0 \times {}^t \left(y - \frac{\sin w}{w}, \frac{1 - \cos w}{w} - z \right).$$

Recall also from Remark 4.1.3 that

$$0 < \det(U^0) = \frac{1}{4w^4} - \frac{4(1 - \cos w) + \sin^2 w}{4w^6} + \frac{(1 - \cos w) \sin w}{w^7} = \frac{w^2}{8640} + \mathcal{O}(w^4)$$

for small $|w|$. With Proposition 5.1 and Remark 4.1.3, this yields the the case $w \neq 0$ of Theorem 2.1, and then (by means of banal computations) Remark 2.3 as well. Note that the uniformity with respect to $(y, z) \in \mathbb{R}^2$ in Proposition 5.1 (already in Proposition 4.3.7) indeed allows to replace (y, z) eventually by $(y, z)/\varepsilon$.

6 The singular case $w = 0$

Let us start again from Lemma 3.2.1 :

$$p_\varepsilon(0; (0, \varepsilon y, \varepsilon z)) = \frac{1}{4\pi^2 \varepsilon^3 \sqrt{2\pi \varepsilon}} \int_{\mathbb{R}^2} P_\varepsilon^0(\xi', \xi) d\xi' d\xi,$$

with

$$\begin{aligned} P_\varepsilon^0(\xi', \xi) &:= \mathbb{E}_0^0 \left[\exp \left(\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\xi' \left[\int_0^1 \cos(\sqrt{\varepsilon} \omega_s) ds - y \right] + \xi \left[\int_0^1 \sin(\sqrt{\varepsilon} \omega_s) ds - z \right] \right) \right) \right] = \\ &\exp \left[\frac{\sqrt{-1}}{\sqrt{\varepsilon}} (\xi' [1 - y] - \xi z) \right] \times \mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 \int_0^1 [\xi \cos(\sqrt{\varepsilon} \tau \omega_s) - \xi' \sin(\sqrt{\varepsilon} \tau \omega_s)] \omega_s ds d\tau \right) \right]. \end{aligned}$$

Let us change ξ' into $\xi'/\sqrt{\varepsilon}$ in (3), so that we instead have :

$$\begin{aligned} &4\pi^2 \sqrt{2\pi} \varepsilon^4 \times p_\varepsilon(0; (0, \varepsilon y, \varepsilon z)) \\ &= \int_{\mathbb{R}^2} \mathbb{E}_0^0 \left[\exp \left(\frac{\sqrt{-1}}{\sqrt{\varepsilon}} \left(\frac{\xi'}{\sqrt{\varepsilon}} \left[\int_0^1 \cos(\sqrt{\varepsilon} \omega_s) ds - y \right] + \xi \left[\int_0^1 \sin(\sqrt{\varepsilon} \omega_s) ds - z \right] \right) \right) \right] d\xi' d\xi \\ &= \int_{\mathbb{R}^2} \varepsilon^{\sqrt{-1} \left(\frac{\xi'}{\varepsilon} [1-y] - \frac{\xi}{\sqrt{\varepsilon}} z \right)} \times \mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 \left[\xi \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} - \xi' \frac{1 - \cos(\sqrt{\varepsilon} \omega_s)}{\varepsilon} \right] ds \right) \right] d\xi' d\xi. \quad (20) \end{aligned}$$

The phase is now $\tilde{A}_\varepsilon(\xi', \xi) := \xi' \tilde{U}_1^\varepsilon + \xi \tilde{U}_2^\varepsilon$, with the phase vector :

$$\tilde{U}^\varepsilon(\omega) := \left(- \int_0^1 \frac{1 - \cos(\sqrt{\varepsilon} \omega_s)}{\varepsilon} ds, \int_0^1 \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} ds \right). \quad (21)$$

The above replaces (4), (6) and (7). We have again to dominate the last term of (20) :

$$\tilde{\mathcal{E}}_\varepsilon(\xi', \xi) := \mathbb{E}_0^0 \left[e^{\sqrt{-1} (\xi' \tilde{U}_1^\varepsilon + \xi \tilde{U}_2^\varepsilon)} \right] \quad (22)$$

(analogue to (5)) by an integrable ε -independent expression.

6.1 Analysis of the phase vector \tilde{U}^ε and second use of [T2]

We here perform the analogue of Sections 4.1 and 4.2 for the singular case $w = 0$. We successively have :

$$\begin{aligned} \tilde{U}^0(\omega) &= \left(-\frac{1}{2} \int_0^1 \omega_s^2 ds, \int_0^1 \omega_s ds \right), \quad (23) \\ \frac{d}{dt} \nabla \tilde{U}_2^\varepsilon(\omega)(t) &= \int_t^1 \cos(\sqrt{\varepsilon} \omega_s) ds - \int_0^1 du \int_u^1 \cos(\sqrt{\varepsilon} \omega_s) ds; \\ \frac{d}{dt} \nabla \tilde{U}_1^\varepsilon(\omega)(t) &= \int_0^1 du \int_u^1 \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} ds - \int_t^1 \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} ds; \end{aligned}$$

$$\mathcal{L}\tilde{U}_1^\varepsilon(\omega) = -\varepsilon^{-1/2} \int_0^1 \sin(\sqrt{\varepsilon} \omega_s) \omega_s ds,$$

$$\frac{d}{dt} \nabla \mathcal{L}\tilde{U}_1^\varepsilon(\omega)(t) = \int_0^t \left[\omega_s \cos(\sqrt{\varepsilon} \omega_s) + \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} \right] ds - \int_0^1 \int_0^t \left[\omega_s \cos(\sqrt{\varepsilon} \omega_s) + \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} \right] ds;$$

and

$$D\tilde{U}^0(\omega) := \begin{pmatrix} \int_0^1 \left[\int_t^1 \omega \right]^2 dt - \left[\int_0^1 dt \int_t^1 \omega \right]^2 & \int_0^1 (t - \frac{1}{2}) \left[\int_t^1 \omega \right] dt \\ \int_0^1 (t - \frac{1}{2}) \left[\int_t^1 \omega \right] dt & 1/12 \end{pmatrix}.$$

Thus the estimates of Lemmas 4.1.6 and 4.1.8 relating to $\|\nabla^k \tilde{U}_2^\varepsilon\|_{H_0^{\otimes k}}$ and $\|\nabla^k \mathcal{L}\tilde{U}_2^\varepsilon\|_{H_0^{\otimes k}}$ remain the same as before (regarding U_2^ε). Whereas for $\|\nabla^k \tilde{U}_1^\varepsilon\|_{H_0^{\otimes k}}$ and $\|\nabla^k \mathcal{L}\tilde{U}_1^\varepsilon\|_{H_0^{\otimes k}}$, a straightforward adaptation provides the following.

Lemma 6.1.1 *We almost surely have :*

$$\|\nabla^k \tilde{U}_1^\varepsilon\|_{H_0^{\otimes k}} \leq 4\sqrt{2} k^{-1} (2\sqrt{\varepsilon})^{k-2}; \quad \|\nabla^k \mathcal{L}\tilde{U}_1^\varepsilon\|_{H_0^{\otimes k}} \leq 4\sqrt{2} (2\sqrt{\varepsilon})^{k-2} \times \left[2 + \varepsilon \int_0^1 \omega^2 \right]^{1/2},$$

for $k \geq 2$, and for $k = 1$:

$$\|\nabla \tilde{U}_1^\varepsilon\|_{H_0} \leq \int_0^1 |\omega|; \quad \|\nabla \mathcal{L}\tilde{U}_1^\varepsilon\|_{H_0} \leq \sqrt{2} \left[\int_0^1 \omega^2 \right]^{1/2}.$$

Thus

$$\|\nabla^k \tilde{A}_\varepsilon(\xi', \xi)\|_{H_0^{\otimes k}} \leq 6 \sqrt{\xi^2 + \xi'^2} (2\sqrt{\varepsilon})^{k-2} \times \left[2 + \varepsilon \int_0^1 \omega^2 \right]^{1/2},$$

for $k \geq 2$, and for $k = 1$:

$$\|\nabla \tilde{A}_\varepsilon(\xi', \xi)\|_{H_0} \leq \sqrt{3} \sqrt{\xi^2 + \xi'^2} \left[\int_0^1 \omega^2 \right]^{1/2}.$$

As a consequence, we directly deduce the following analogue of Lemma 4.1.9.

Lemma 6.1.2 *For any positive r, ε we almost surely have :*

$$M_1[\tilde{U}_1^\varepsilon, r] + M_1[\tilde{U}_2^\varepsilon, r] < 3 + 8r^2(1 + \varepsilon) e^{4r\varepsilon} + \int_0^1 \omega^2; \quad M_2[\tilde{U}_1^\varepsilon, r] + M_2[\tilde{U}_2^\varepsilon, r] < 8r^2(1 + \varepsilon) e^{4r\varepsilon};$$

$$M_1[\mathcal{L}\tilde{U}_1^\varepsilon, r] + M_1[\mathcal{L}\tilde{U}_2^\varepsilon, r] < 8r(2r + 1) e^{4r\varepsilon} \left[2 + \varepsilon \int_0^1 \omega^2 \right] + 2 \int_0^1 \omega^2;$$

$$M_2[\mathcal{L}\tilde{U}_1^\varepsilon, r] + M_2[\mathcal{L}\tilde{U}_2^\varepsilon, r] < (16 + 8\varepsilon) r^2 e^{4r\varepsilon} \left[2 + \varepsilon \int_0^1 \omega^2 \right].$$

Then this entails :

$$L_2[\tilde{U}^\varepsilon, r] \leq 8r^2(1 + \varepsilon) e^{4r\varepsilon} + (16 + 8\varepsilon) r^2 e^{4r\varepsilon} \left[2 + \varepsilon \int_0^1 \omega^2 \right] + 1 + \int_0^1 \omega^2,$$

and (to replace (13)) :

$$\begin{aligned}
1 \leq L_1[\tilde{U}^\varepsilon, r] &\leq 4 + 8r^2(1 + \varepsilon)e^{4r\varepsilon} + 8r(2r + 1)e^{4r\varepsilon} \left[2 + \varepsilon \int_0^1 \omega^2 \right] + 4 \int_0^1 \omega^2 \\
&\leq \mathcal{O}(1) + (4 + \mathcal{O}(\varepsilon))(\omega^*)^2 =: \tilde{\Omega}_\varepsilon.
\end{aligned} \tag{24}$$

Then we proceed as in Section 4.2, in order to apply (2.1) in Theorem 2.1 of [T2] again, of course with $\tilde{U}^\varepsilon \equiv \tilde{q}$ instead of U^ε . As in (11) and for the same reason, we again have

$$\min \left\{ \left\| \nabla \tilde{q}^{(z)} \right\|_{H_0}^2 \mid |z| = 1 \right\} \geq \frac{\det(D\tilde{U}^\varepsilon)}{\text{tr}(D\tilde{U}^\varepsilon)},$$

so that for some $C > 0$, any $r \in]1, e[$ and any $\xi^2 + \xi'^2 > 1$, we again have (12):

$$\left| \mathbb{E}_0^0 \left[e^{\sqrt{-1} (\xi' \tilde{U}_1^\varepsilon + \xi \tilde{U}_2^\varepsilon)} \right] \right|^2 \leq C \mathbb{E}_0^0 \left[\exp \left(\frac{L_2[\tilde{U}^\varepsilon, r]}{L_1[\tilde{U}^\varepsilon, 1]^3} \right) \right] \times \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(D\tilde{U}^\varepsilon)/\text{tr}(D\tilde{U}^\varepsilon)}{147 C L_1[\tilde{U}^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right],$$

and then

$$\mathbb{E}_0^0 \left[\exp \left(\frac{L_2[\tilde{U}^\varepsilon, r]}{L_1[\tilde{U}^\varepsilon, 1]^3} \right) \right] \leq \mathbb{E}_0^0 \left[e^{L_2[\tilde{U}^\varepsilon, r]} \right] \leq \sqrt{C} \mathbb{E}_0^0 \left[e^{(1+\varepsilon \log C) \int_0^1 \omega^2} \right] = \sqrt{\frac{\sqrt{2(1+\varepsilon \log C)}}{\sin(\sqrt{2(1+\varepsilon \log C)}}} \leq 2$$

for small enough ε . See Corollary 7.2 below for the Laplace transform of the law of $\int_0^1 \omega^2$ under \mathbb{P}_0^0 . Therefore the analogue of (15) holds:

$$\left| \mathbb{E}_0^0 \left[e^{\sqrt{-1} (\xi' \tilde{U}_1^\varepsilon + \xi \tilde{U}_2^\varepsilon)} \right] \right|^2 \leq 2C \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(D\tilde{U}^\varepsilon)/\text{tr}(D\tilde{U}^\varepsilon)}{147 C L_1[\tilde{U}^\varepsilon, 1]} \sqrt{\xi^2 + \xi'^2} \right) \right],$$

and then, owing to (22) and (24):

$$\left| \tilde{\mathcal{E}}_\varepsilon(\xi', \xi) \right|^2 \leq 2C \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(D\tilde{U}^\varepsilon)}{\text{tr}(D\tilde{U}^\varepsilon) \tilde{\Omega}_\varepsilon} \times \frac{\sqrt{\xi^2 + \xi'^2}}{147 C} \right) \right]. \tag{25}$$

6.2 Domination of the decreasing integral in the singular case

Here we perform the analogue of Section 4.3 for the singular case $w = 0$. We have to estimate the new key variance (determinant of the Malliavin matrix) from below, and to dominate the related expected value. Proceeding as for Proposition 4.3.1, let us denote by $\tilde{\lambda}_\varepsilon$ the (almost surely positive) lowest eigenvalue of $D\tilde{U}^\varepsilon$. Then on the one hand we have the following analogue of (16)(17), for any deterministic positive R :

$$\begin{aligned}
&\mathbb{E}_0^0 \left[e^{-\frac{\det(D\tilde{U}^\varepsilon)}{\text{tr}(D\tilde{U}^\varepsilon) \tilde{\Omega}_\varepsilon} \times R} \right] \leq \mathbb{E}_0^0 \left[\exp \left(- \frac{\tilde{\lambda}_\varepsilon R}{2 [\mathcal{O}(1) + (4 + \mathcal{O}(\varepsilon))(\omega^*)^2]} \right) \right] \\
&\leq \mathbb{P}_0^0(\omega^* > R^{1/3}/C) + \mathbb{E}_0^0 \left[\exp \left(\frac{-\tilde{\lambda}_\varepsilon R}{\mathcal{O}(1) + (4 + \mathcal{O}(\varepsilon))C^{-2}R^{2/3}} \right) 1_{\{\omega^* \leq R^{1/3}/C\}} \right]
\end{aligned}$$

$$\leq 2e^{-2C^{-2}R^{2/3}} + \mathbb{E}_0^0 \left[e^{-\tilde{\lambda}_\varepsilon R^{1/3}} 1_{\{\omega^* \leq R^{1/3}\}} \right] \quad (\text{for } 0 \leq \varepsilon \leq 1 \text{ and large enough } R). \quad (26)$$

On the other hand, we have :

$$\langle v, D\tilde{U}^\varepsilon v \rangle = \text{Var}_{[0,1]} \left[\int_{\bullet}^1 \left[v_1 \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} + v_2 \cos(\sqrt{\varepsilon} \omega_s) \right] ds \right] \quad \text{for any } v \in \mathbb{S}^1,$$

so that, setting

$$\begin{aligned} V_\varepsilon^\theta(\omega) &:= \text{Var}_{[0,1]} \left[\sin \theta \int_{\bullet}^1 \cos(\sqrt{\varepsilon} \omega_s) + \cos \theta \int_{\bullet}^1 \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} \right] \\ &= \int_{0 < t < u < 1} \left[\sin \theta \int_t^u \cos(\sqrt{\varepsilon} \omega_s) + \cos \theta \int_t^u \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} \right]^2 dt du \quad \text{according to (19),} \end{aligned} \quad (27)$$

we have :

$$\tilde{\lambda}_\varepsilon = \inf_{\theta} V_\varepsilon^\theta. \quad (28)$$

According to (25), our aim is to establish that

$$\int_1^\infty \sup_{0 \leq \varepsilon \leq 1} \sqrt{\mathbb{E}_0^0 \left[e^{-\tilde{\lambda}_\varepsilon R^{1/3}} 1_{\{\omega^* \leq R^{1/3}\}} \right]} R dR < \infty. \quad (29)$$

6.2.1 Getting the extremum over θ out of the expectation \mathbb{E}_0^0

We must now obtain the analogue of Proposition 4.3.5 relating to Variance V_ε^θ .

Note that by symmetry it is enough to consider $\theta \in [0, \pi[$, and that using (19) we easily have: for any $\theta, \theta', \varepsilon, \omega$,

$$|V_\varepsilon^\theta(\omega) - V_\varepsilon^{\theta'}(\omega)| \leq |\theta - \theta'| \times \frac{1 + (\omega^*)^2}{6}. \quad (30)$$

We now use a nice idea of ([H], Lemma 4.7), which consists in getting an extremum out of a probability, by means of a simple counting argument. Namely, for any positive t consider the angles kt , for $\mathbb{N} \ni k < \pi/t$. Since by (30), restricting on the event $\{\omega^* \leq R^{1/3}\}$ we have $|V_\varepsilon^\theta - V_\varepsilon^{kt}| \leq |\theta - kt| \frac{1+R^{2/3}}{6} \leq |\theta - kt| \frac{R^{2/3}}{4}$ (for $R \geq 3$), we obtain

$\inf_{\theta} V_\varepsilon^\theta \geq \inf_k V_\varepsilon^{kt} - t \frac{R^{2/3}}{4}$, and then, on the event $\{\omega^* \leq R^{1/3}\}$:

$$\left\{ \inf_{\theta} V_\varepsilon^\theta < R^{-1/3} s \right\} \subset \left\{ \inf_k V_\varepsilon^{kt} < R^{-1/3} s + \frac{R^{2/3}}{4} t \right\} \subset \bigcup_{0 \leq k < \pi/t} \left\{ V_\varepsilon^{kt} < R^{-1/3} s + \frac{R^{2/3}}{4} t \right\}.$$

Choosing $t = \pi s/R$, we thus obtain :

$$\mathbb{P}_0^0 \left(\tilde{\lambda}_\varepsilon < R^{-1/3} s, \omega^* \leq R^{1/3} \right) \leq \frac{R}{s} \sup_{\theta} \mathbb{P}_0^0 \left(V_\varepsilon^\theta < 2 R^{-1/3} s \right).$$

Hence,

$$\mathbb{E}_0^0 \left[e^{-\tilde{\lambda}_\varepsilon R^{1/3}} 1_{\{\omega^* \leq R^{1/3}\}} \right] = \int_0^\infty \mathbb{P}_0^0 \left(\tilde{\lambda}_\varepsilon < R^{-1/3} s, \omega^* \leq R^{1/3} \right) e^{-s} ds$$

$$\leq R \int_0^\infty \sup_\theta \mathbb{P}_0^\theta(V_\varepsilon^\theta < 2 R^{-1/3} s) e^{-s} \frac{ds}{s} = R \int_0^\infty \sup_\theta \mathbb{P}_0^\theta(V_\varepsilon^\theta < 2 s) e^{-R^{1/3} s} \frac{ds}{s},$$

whence

$$\mathbb{E}_0^\theta \left[e^{-\tilde{\lambda}_\varepsilon R^{1/3}} 1_{\{\omega^* \leq R^{1/3}\}} \right] \leq R^{2/3} e^{-R^{1/3}} + R \int_0^1 \sup_\theta \mathbb{P}_0^\theta(V_\varepsilon^\theta < 2 s) e^{-R^{1/3} s} \frac{ds}{s}.$$

As a consequence, the estimate (29) will follow now from

$$\int_1^\infty \sup_{0 \leq \varepsilon \leq 1} \sqrt{\int_0^1 \sup_\theta \mathbb{P}_0^\theta(V_\varepsilon^\theta < 2 s) e^{-R^{1/3} s} \frac{ds}{s}} R^{3/2} dR < \infty,$$

or equivalently,

$$\int_1^\infty \sup_{0 \leq \varepsilon \leq 1} \sqrt{\int_0^1 \sup_\theta \mathbb{P}_0^\theta(V_\varepsilon^\theta < 2 s) e^{-R s} \frac{ds}{s}} R^{13/2} dR < \infty. \quad (31)$$

6.2.2 Estimating the Variance from below for $w = 0$

We here establish the following substitute for Proposition 4.3.5, with the big difference that in the present degenerate case $w = 0$, an uniform estimate of the variance from below can no longer be sufficient. Thus a somewhat more subtle proof is here necessary.

Proposition 6.2.3 *The following estimate holds: for any $x \in [0, 1]$ and $p \geq 2$, there exists an explicit positive finite constant C_p such that we have:*

$$\sup_\theta \mathbb{P}_0^\theta \left(\inf_{0 \leq \varepsilon \leq 1} V_\varepsilon^\theta < x \right) \leq C_p \mathbb{E}_0(S_p^{2p}) x^{(2p-3-3/p)/4} + \min \left\{ 2, \frac{4\sqrt{\pi} x^{-1/8} e^{-\frac{\pi}{196} x^{-1/4}}}{7 \left(1 - e^{-\frac{2\pi}{49} x^{-1/4}} \right)} \right\}.$$

Proof It is enough to consider $0 \leq \theta < \pi$. Again let $\omega^* := \max_{0 \leq s \leq 1} |\omega_s|$.

(i) Consider first the stopping time $T := \min \{1, \inf \{s > 0 \mid |\omega_s| = \frac{1}{2} |\sin \theta|\}\}$. Then on the one hand for $0 \leq s \leq T$ we have

$$\sin \theta \cos(\sqrt{\varepsilon} \omega_s) + \cos \theta \frac{\sin(\sqrt{\varepsilon} \omega_s)}{\sqrt{\varepsilon}} \geq \frac{1}{2} |\sin \theta| \cos\left(\frac{1}{2} \sqrt{\varepsilon} |\sin \theta|\right) - \frac{1}{2} \cos \theta |\sin \theta| \geq \frac{1}{2} |\sin \theta|,$$

so that by (27) we have:

$$V_\varepsilon^\theta(\omega) \geq \int_{0 < t < u < T} \left[\int_t^u \frac{1}{2} |\sin \theta| ds \right]^2 dt du = \frac{T^2 \sin^2 \theta}{48},$$

and on the other hand, by Remark 4.3.4 we have: either $T = 1$ (on $\{\omega^* < \frac{1}{2} |\sin \theta|\}$) or

$$\frac{1}{2} |\sin \theta| = |\omega_T - \omega_0| \leq S_p(\omega) \times |T(\omega)|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}},$$

so that \mathbb{P}_0^θ -almost surely

$$V_\varepsilon^\theta \geq \frac{|\sin \theta|^{\frac{4p-2-2/p}{p-1-1/p}}}{48} (2S_p)^{\frac{-2p}{p-1-1/p}}.$$

(ii) Suppose then that $\sin \theta < 1/9$. We distinguish two sub-cases, as follows.

(iii1) Consider first the case $\omega^* > \sqrt{1/\varepsilon}$, and thereon the stopping times :

$$T_1 := \inf\{s > 0 \mid |\sin(\sqrt{\varepsilon}\omega_s)| = \sin 1\} \quad \text{and} \quad T_2 := \inf\{s > T_1 \mid |\sin(\sqrt{\varepsilon}\omega_s)| = \tfrac{1}{2} \sin 1\}.$$

Then for any $s \in [T_1, T_2]$ we have $\left| \frac{\sin(\sqrt{\varepsilon}\omega_s)}{\sqrt{\varepsilon}} \right| \geq \frac{\sin 1}{2\sqrt{\varepsilon}}$, which entails

$$\left| \cos \theta \frac{\sin(\sqrt{\varepsilon}\omega_s)}{\sqrt{\varepsilon}} \pm \sin \theta \cos(\sqrt{\varepsilon}\omega_s) \right| \geq \sqrt{80/81} \frac{\sin 1}{2\sqrt{\varepsilon}} - \frac{1}{9} > \frac{2}{7\sqrt{\varepsilon}},$$

so that

$$V_\varepsilon^\theta \geq \int_{T_1 < t < u < T_2} [2(u-t)/7]^2 dt du = \frac{(T_2 - T_1)^2}{147}.$$

Since moreover (as previously)

$$\frac{1}{2\sqrt{2}} < \tfrac{1}{2} \sin 1 = |\omega_{T_2} - \omega_{T_1}| \leq S_p(\omega) \times |T_2(\omega) - T_1(\omega)|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}},$$

we obtain: \mathbb{P}_0^0 -almost surely, for any $p \geq 2$:

$$V_\varepsilon^\theta \geq \frac{1}{150} \left(2\sqrt{2} S_p\right)^{\frac{-2p}{p-1-1/p}} 1_{\{9|\sin \theta| < 1 < \sqrt{\varepsilon}\omega^*\}} \geq \frac{1}{48} (5S_p)^{\frac{-2p}{p-1-1/p}} 1_{\{9|\sin \theta| < 1 < \sqrt{\varepsilon}\omega^*\}}.$$

(iii2) Consider then the case $7|\sin \theta| \leq \omega^* \leq \sqrt{1/\varepsilon}$, and thereon the stopping times :

$$T_1 := \inf\{s > 0 \mid |\omega_s| = \omega^*\} \quad \text{and} \quad T_2 := \inf\{s > T_1 \mid |\omega_s| = 4\omega^*/5\}.$$

Then for any $s \in [T_1, T_2]$ we have :

$$\left| \frac{\sin(\sqrt{\varepsilon}\omega_s)}{\sqrt{\varepsilon}} \right| \geq |\omega_s| \times \left(1 - \frac{\varepsilon(\omega^*)^2}{6}\right) \geq \frac{2}{3}\omega^*,$$

whence

$$\left| \cos \theta \frac{\sin(\sqrt{\varepsilon}\omega_s)}{\sqrt{\varepsilon}} \pm \sin \theta \cos(\sqrt{\varepsilon}\omega_s) \right| \geq \sqrt{80/81} \frac{2}{3}\omega^* - |\sin \theta| > \omega^*/2,$$

so that

$$V_\varepsilon^\theta \geq \int_{T_1 < t < u < T_2} [(u-t)\omega^*/2]^2 dt du = \frac{(T_2 - T_1)^2}{48} (\omega^*)^2.$$

Since moreover (as previously) $\frac{\omega^*}{5} = |\omega_{T_2} - \omega_{T_1}| \leq S_p(\omega) \times |T_2(\omega) - T_1(\omega)|^{\frac{1}{2} - \frac{1}{2p} - \frac{1}{2p^2}}$, we obtain: \mathbb{P}_0^0 -almost surely,

$$V_\varepsilon^\theta \geq \frac{(\omega^*)^{\frac{4p-2-2/p}{p-1-1/p}}}{48} (5S_p)^{\frac{-2p}{p-1-1/p}} 1_{\{|\sin \theta| < 1/9, 7|\sin \theta| < \omega^* \leq \sqrt{1/\varepsilon}\}}.$$

(iii) Summing up the above cases (i), (ii1) and (ii2), so far we have: for any $p \geq 2$,

$$\inf_{0 \leq \varepsilon \leq 1} V_\varepsilon^\theta \geq \frac{1}{48} \left(\max\{|\sin \theta|, \min\{\omega^*, 1\}\} \times 1_{\{7|\sin \theta| \leq \omega^*\}} \right)^{\frac{4p-2-2/p}{p-1-1/p}} \times (5S_p)^{\frac{-2p}{p-1-1/p}}.$$

This entails: for any $x \in [0, 1]$ and any $p \geq 2$,

$$\begin{aligned} \mathbb{P}_0^0 \left(\inf_{0 \leq \varepsilon \leq 1} V_\varepsilon^\theta < x \right) &\leq \mathbb{P}_0^0 \left(\omega^* < 7|\sin \theta|, (\sin^2 \theta)^{2p-1-1/p} < (5S_p)^{2p} (48x)^{p-1-1/p} \right) \\ &\quad + \mathbb{P}_0^0 \left(\omega^* \geq 7|\sin \theta|, (\min\{\omega^*, 1\})^{4p-2-2/p} < (5S_p)^{2p} (48x)^{p-1-1/p} \right) \\ &\leq 1_{\{|\sin \theta| > x^{1/8}\}} \mathbb{P}_0^0 \left(x^{(2p-1-1/p)/4} < (5S_p)^{2p} (48x)^{p-1-1/p} \right) + 2 \times 1_{\{|\sin \theta| \leq x^{1/8}\}} \mathbb{P}_0^0 (\omega^* < 7x^{1/8}) \\ &\quad + \mathbb{P}_0^0 \left((x^{1/8})^{4p-2-2/p} < (5S_p)^{2p} (48x)^{p-1-1/p} \right), \end{aligned}$$

so that for any $p \geq 2$ we obtain:

$$\mathbb{P}_0^0 \left(\inf_{0 \leq \varepsilon \leq 1} V_\varepsilon^\theta < x \right) \leq 2(48)^{p-1-1/p} \mathbb{E}_0^0((5S_p)^{2p}) x^{(2p-3-3/p)/4} + 2\mathbb{P}_0^0(\omega^* < 7x^{1/8}). \quad (32)$$

(iv) Now, according to ([B-O] (4.12)) and to the Poisson formula (see for example (38) in [B-P-Y]), for any positive y we have

$$\mathbb{P}_0^0(\omega^* < y) = \sum_{n \in \mathbb{Z}} (-1)^n e^{-2n^2 y^2} = \frac{2\sqrt{\pi}}{y} \sum_{n \in \mathbb{N}} \exp\left(-\frac{(2n+1)^2 \pi^2}{4y^2}\right),$$

or equivalently

$$\mathbb{P}_0^0(\omega^* < \sqrt{\pi} y) = \frac{2}{y} e^{-y^{-2}/4} \sum_{n \in \mathbb{N}} e^{-n(n+1)/y^2} < \frac{2e^{-y^{-2}/4}}{x(1 - e^{-2/y^2})}. \quad (33)$$

By (32) and (33), setting $C_p := 2 \times 5^{2p} \times (48)^{p-1-1/p}$, we finally obtain: for $0 \leq x \leq 1$,

$$\sup_{\theta} \mathbb{P}_0^0 \left(\inf_{0 \leq \varepsilon \leq 1} V_\varepsilon^\theta < x \right) \leq C_p \mathbb{E}_0^0(S_p^{2p}) x^{(2p-3-3/p)/4} + \min \left\{ 2, \frac{4\sqrt{\pi} x^{-1/8} e^{-\frac{\pi}{196} x^{-1/4}}}{7 \left(1 - e^{-\frac{2\pi}{49} x^{-1/4}} \right)} \right\}. \quad \diamond$$

6.2.4 Key domination relating to the singular case $w = 0$

The following crucial estimate is the singular analogue of Proposition 4.3.1.

Proposition 6.2.5 *For any fixed positive C , we have*

$$\sup_{0 \leq \varepsilon \leq 1} \mathbb{E}_0^0 \left[\exp \left(- \frac{\det(D\tilde{U}^\varepsilon)}{\text{tr}(D\tilde{U}^\varepsilon) \tilde{\Omega}^\varepsilon} \times \frac{\sqrt{\xi^2 + \xi'^2}}{147C} \right) \right] \in L^{1/2}(\mathbb{R}^2, d\xi' d\xi).$$

Proof Recall that according to (26), we have to show that (29) holds, which by Section 6.2.1 amounts to showing that (31) holds. Now, applying Proposition 6.2.3 with $p \geq 34$ we obtain :

$$\begin{aligned}
& \left(\int_1^\infty \sup_{0 \leq \varepsilon \leq 1} \sqrt{\int_0^1 \sup_\theta \mathbb{P}_0^0(V_\varepsilon^\theta < 2s) e^{-Rs} \frac{ds}{s}} R^{13/2} dR \right)^2 \\
& \leq \int_1^\infty \sup_{0 \leq \varepsilon \leq 1} \left[\int_0^1 \sup_\theta \mathbb{P}_0^0(V_\varepsilon^\theta < 2s) e^{-Rs} \frac{ds}{s} \right] R^{15} dR \times \int_1^\infty R^{-2} dR \\
& \leq \int_1^\infty \left[\int_0^1 C_p \mathbb{E}_0^0(S_p^{2p}) (2s)^{(2p-3-3/p)/4} e^{-Rs} \frac{ds}{s} \right] R^{15} dR \\
& \quad + \int_1^\infty \left[\int_0^1 \min \left\{ 2, \frac{4\sqrt{\pi} (2s)^{-1/8} e^{-\frac{\pi}{196}(2s)^{-1/4}}}{7 \left(1 - e^{-\frac{2\pi}{49}(2s)^{-1/4}} \right)} \right\} e^{-Rs} \frac{ds}{s} \right] R^{15} dR \\
& = C_p \mathbb{E}_0^0(S_p^{2p}) 2^{(2p-3-3/p)/4} \int_1^\infty \left[\int_0^R s^{(2p-3-3/p)/4-1} e^{-s} ds \right] R^{15-(2p-3-3/p)/4} dR \\
& \quad + \int_0^1 \left[\int_1^\infty e^{-Rs} R^{15} dR \right] \min \left\{ 2, \frac{4\sqrt{\pi} (2s)^{-1/8} e^{-\frac{\pi}{196}(2s)^{-1/4}}}{7 \left(1 - e^{-\frac{2\pi}{49}(2s)^{-1/4}} \right)} \right\} \frac{ds}{s} \\
& = C_p \mathbb{E}_0^0(S_p^{2p}) 2^{(2p-3-3/p)/4} \int_0^\infty \left[\int_s^\infty R^{15-(2p-3-3/p)/4} dR \right] s^{(2p-3-3/p)/4-1} e^{-s} ds \\
& \quad + \int_0^1 \left[\int_s^\infty e^{-R} R^{15} dR \right] \min \left\{ 2, \frac{4\sqrt{\pi} (2s)^{-1/8} e^{-\frac{\pi}{196}(2s)^{-1/4}}}{7 \left(1 - e^{-\frac{2\pi}{49}(2s)^{-1/4}} \right)} \right\} s^{-17} ds \\
& \leq C_p \mathbb{E}_0^0(S_p^{2p}) \frac{2^{(2p+5-3/p)/4}}{2p-67-3/p} \int_0^\infty e^{-s} s^{15} ds + 2^{17} (15)! \int_0^2 \min \left\{ 1, \frac{2\sqrt{\pi} s^{-1/2} e^{-\frac{\pi}{196} s^{-1}}}{7 \left(1 - e^{-\frac{2\pi}{49} s^{-1}} \right)} \right\} s^{-65} ds
\end{aligned}$$

which (owing to Remark 4.3.4) is clearly finite, for example for $p = 34$. \diamond

6.3 Intermediate small-time equivalent for p_ε in the case $w = 0$

The preceding estimate entails the following wanted approximation result, which is the singular analogue of Propositions 4.3.7 and 5.1.

Proposition 6.3.1 *Uniformly with respect to $(y, z) \in \mathbb{R}^2$, as $\varepsilon \searrow 0$ we have*

$$p_\varepsilon(0; (0, y, z)) = \frac{1 + o(1)}{4\pi^2 \varepsilon^4 \sqrt{2\pi}} \times \mathcal{I}_\varepsilon(y, z),$$

with

$$\mathcal{I}_\varepsilon(y, z) := \int_{\mathbb{R}^2} \exp \left[\sqrt{-1} \left(\frac{\xi'}{\varepsilon^2} (\varepsilon - y) - \frac{\xi}{\varepsilon^{3/2}} z \right) \right] \tilde{\mathcal{E}}_0(\xi', \xi) d\xi' d\xi,$$

and $\tilde{\mathcal{E}}_0(\xi', \xi) = \mathbb{E}_0^0 \left[\exp \left(\sqrt{-1} \int_0^1 [\xi \omega_s - \xi' \omega_s^2 / 2] ds \right) \right] \in L^1(\mathbb{R}^2, d\xi' d\xi).$

Proof By (20), (21) and (22) (in the beginning of Section 6), we have

$$p_\varepsilon(0; (0, \varepsilon y, \varepsilon z)) = \frac{1}{4\pi^2 \varepsilon^4 \sqrt{2\pi}} \int_{\mathbb{R}^2} P_\varepsilon^0(\xi', \xi) d\xi' d\xi,$$

with

$$P_\varepsilon^0(\xi', \xi) = \exp \left[\sqrt{-1} \left(\frac{\xi'}{\varepsilon} (1 - y) - \frac{\xi}{\sqrt{\varepsilon}} z \right) \right] \times \tilde{\mathcal{E}}_\varepsilon(\xi', \xi).$$

By (25) and Proposition 6.2.5 we have $\sup_{0 \leq \varepsilon \leq 1} |\tilde{\mathcal{E}}_\varepsilon(\xi', \xi)| \in L^1(\mathbb{R}^2, d\xi' d\xi)$. This provides the wanted domination of $P_\varepsilon^0(\xi', \xi)$, which allows to apply the Lebesgue theorem with respect to $d\xi' d\xi$, using the continuity of $\tilde{\mathcal{E}}_\varepsilon(\xi', \xi)$ at $\varepsilon = 0$ and then (23). As $\tilde{\mathcal{E}}_\varepsilon(\xi', \xi)$ does not depend on (y, z) , this convergence is uniform with respect to $(y, z) \in \mathbb{R}^2$, which we can finally replace by $(y, z)/\varepsilon$. \diamond

7 Computation of a quadratic Laplace transform

Proposition 6.3.1 yields the wanted equivalent for $p_\varepsilon(0; (0, y, z))$ in terms of the Laplace transform $\tilde{\mathcal{E}}_0(\xi', \xi)$ of a quadratic functional of the Brownian bridge.

We here perform the computation of a slightly more general Brownian bridge Laplace transform. This was already needed in Section 6.1, and will be crucial in the forthcoming section 8. The principle of this type of computation goes back to Yor [Y].

Proposition 7.1 *We have* $\mathbb{E}_0^0 \left[\exp \left(\int_0^1 [\alpha_s \omega_s + \gamma_s \omega_s^2] ds \right) \right] =$

$$= \frac{\exp \left[\frac{1}{2} \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds - \left(\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds \right)^2 \left(2 \int_0^1 e^{2 \int_1^s g} ds \right)^{-1} - \frac{1}{2} \int_0^1 g \right]}{\sqrt{\int_0^1 e^{2 \int_1^s g} ds}},$$

where α_s, γ_s are real deterministic, $\gamma_s \leq 0$, and g solves the Riccati equation $g' = g^2 + 2\gamma$ (equivalent to the linear equation $\frac{d^2}{ds^2} \exp(-\int_0^s g) = -2\gamma_s \exp(-\int_0^s g)$) a.e. on $[0, 1]$.

Proof Set $J^v := \mathbb{E}_0^v \left[\exp \left(\int_0^1 [\alpha_s \omega_s + \gamma_s \omega_s^2] ds \right) \right]$, so that

$$\int_{\mathbb{R}} \frac{e^{-(\theta+1)v^2/2}}{\sqrt{2\pi}} J^v dv = \mathbb{E}_0 \left[\exp \left(\int_0^1 [\alpha_s \omega_s + \gamma_s \omega_s^2] ds - \frac{1}{2} \theta \omega_1^2 \right) \right] =: Y_\theta. \quad (34)$$

Consider the exponential martingale defined by

$$M_s^g := \exp\left(-\int_0^s g_\tau \omega_\tau d\omega_\tau - \frac{1}{2} \int_0^s g_\tau^2 \omega_\tau^2 d\tau\right) = \exp\left(\frac{1}{2} \int_0^s g - \frac{1}{2} g_s \omega_s^2 + \frac{1}{2} \int_0^s (g'_\tau - g_\tau^2) \omega_\tau^2 d\tau\right).$$

Denoting by \mathbb{P}^g the new probability law having M_s^g as density on \mathcal{F}_s with respect to \mathbb{P}_0 , we have :

$$\begin{aligned} Y_\theta e^{\int_0^1 f/2} &= \mathbb{E}^g \left[\exp\left(\frac{1}{2}(g_1 - \theta)\omega_1^2 + \int_0^1 \left[\alpha_s \omega_s + \left(\gamma_s - \frac{1}{2}(g'_s - g_s^2)\right) \omega_s^2\right] ds\right) \right] \\ &= \mathbb{E}^g \left[\exp\left(\frac{1}{2}(g_1 - \theta)\omega_1^2 + \int_0^1 \alpha_s \omega_s ds\right) \right], \end{aligned} \quad (35)$$

by taking g almost everywhere solving the Ricatti equation $g' = g^2 + 2\gamma$ (equivalent to the linear equation $\exp[-\int g]'' = -2\gamma \exp[-\int g]$).

On the other hand, the Girsanov formula provides a $(\mathbb{P}^g, \mathcal{F}_s)$ Brownian motion B such that $\omega_s = B_s - \int_0^s g_\tau \omega_\tau d\tau$, and then $\omega_s = \int_0^s \exp\left(\int_s^\tau g\right) dB_\tau$. Hence, for any real r :

$$\begin{aligned} \mathbb{E}^g \left[\left(r \omega_1 + \int_0^1 \alpha_s \omega_s ds \right)^2 \right] &= \mathbb{E}^g \left[\left(\int_0^1 \left[r + \int_s^1 \alpha \right] d\omega_s \right)^2 \right] \\ &= \mathbb{E}^g \left[\left(\int_0^1 \left[r + \int_s^1 \alpha \right] \left[dB_s - g_s \left(\int_0^s e^{\int_s^\tau g} dB_\tau \right) ds \right] \right)^2 \right] \\ &= \mathbb{E}^g \left[\left(\int_0^1 \left[r + \int_s^1 \alpha - \int_s^1 \left(r + \int_\tau^1 \alpha \right) g_\tau e^{\int_\tau^s g} d\tau \right] dB_s \right)^2 \right] \\ &= \int_0^1 \left[r e^{\int_1^s g} + \int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right]^2 ds. \end{aligned}$$

This yields the covariance matrix of the \mathbb{P}^g -Gaussian variable $\left(\omega_1, \int_0^1 \alpha_s \omega_s ds \right)$, namely

$$K = \begin{pmatrix} \int_0^1 e^{2 \int_1^s g} ds & \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds \\ \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds & \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds \end{pmatrix},$$

whence the density of $\left(\omega_1, \int_0^1 \alpha_s \omega_s ds \right)$ with respect to \mathbb{P}^g . Thus

$$\begin{aligned} \mathbb{E}^g \left[\exp\left(\int_0^1 \alpha_s \omega_s ds - r \omega_1^2\right) \right] &= \int_{\mathbb{R}^2} e^{v-r u^2} \exp\left[\frac{-1}{2 \det K} \left(u^2 \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds \right. \right. \\ &\quad \left. \left. - 2uv \int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds + v^2 \int_0^1 e^{2 \int_1^s g} ds \right) \right] \frac{du dv}{2\pi \sqrt{\det K}} = \\ &= \int_{\mathbb{R}^2} \exp\left[\frac{-1}{2} \left(\left[\frac{\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds}{\det K} + 2r \right] u^2 + \left[\frac{\int_0^1 e^{2 \int_1^s g} ds}{\det K} \right] v^2 - 2v \right) \right] du dv \end{aligned}$$

$$\begin{aligned}
& -2 \left[\frac{\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds}{\det K} uv \right] \frac{du dv}{2\pi \sqrt{\det K}} \\
& = \left(1 + 2r \int_0^1 e^{2 \int_1^s g} ds \right)^{-1/2} \times \exp \left(\frac{\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds + 2r \det K}{2 \left(1 + 2r \int_0^1 e^{2 \int_1^s g} ds \right)} \right) \quad (36)
\end{aligned}$$

(by a classical Gaussian computation). Finally by (34), (35) and (36) we have

$$\begin{aligned}
& \mathbb{E}_0^0 \left[\exp \left(\int_0^1 [\alpha_s \omega_s + \gamma_s \omega_s^2] ds \right) \right] = J^0 = \lim_{\theta \rightarrow \infty} \sqrt{\theta + 1} \int_{\mathbb{R}} \frac{e^{-(\theta+1)v^2/2}}{\sqrt{2\pi}} J^v dv \\
& = \lim_{\theta \rightarrow \infty} \sqrt{\theta + 1} Y_\theta = \lim_{r \rightarrow \infty} \sqrt{2r} \times \mathbb{E}^g \left[\exp \left(\int_0^1 \alpha_s \omega_s ds - r \omega_1^2 \right) \right] \times e^{-\int_0^1 g/2} \\
& = \left(\int_0^1 e^{2 \int_1^s g} ds \right)^{-1/2} \times \exp \left(\frac{\det K}{2 \int_0^1 e^{2 \int_1^s g} ds} - \frac{1}{2} \int_0^1 g \right) \\
& = \frac{\exp \left(\frac{(\int_0^1 e^{2 \int_1^s g} ds) \left(\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right)^2 ds \right) - \left(\int_0^1 \left(\int_s^1 e^{\int_\tau^s g} \alpha_\tau d\tau \right) e^{\int_1^s g} ds \right)^2}{2 \int_0^1 e^{2 \int_1^s g} ds} - \frac{1}{2} \int_0^1 g \right)}{\sqrt{\int_0^1 e^{2 \int_1^s g} ds}} \cdot \diamond
\end{aligned}$$

Corollary 7.2 For any real constants α and γ , we have

$$\mathbb{E}_0^0 \left[\exp \left(\int_0^1 [\alpha \omega_s - \frac{1}{2} \gamma^2 \omega_s^2] ds \right) \right] = \sqrt{\frac{\gamma}{\text{sh } \gamma}} \exp \left[\frac{\alpha^2}{\gamma^3} \left(\frac{\gamma}{2} - \text{th} \left(\frac{\gamma}{2} \right) \right) \right].$$

Proof We apply Proposition 7.1 with $\gamma_s \equiv -\gamma^2/2$, so that we can take $g \equiv \gamma$, yielding:

$$\begin{aligned}
& \mathbb{E}_0^0 \left[\exp \left(\int_0^1 [\alpha \omega_s - \frac{1}{2} \gamma^2 \omega_s^2] ds \right) \right] \\
& = \frac{\exp \left[\frac{\alpha^2}{2} \int_0^1 \left(\frac{1-e^{\gamma(s-1)}}{\gamma} \right)^2 ds - \left(\int_0^1 \left(\frac{1-e^{\gamma(s-1)}}{\gamma} \right) e^{\gamma(s-1)} ds \right)^2 \frac{\alpha^2 \gamma}{1-e^{-2\gamma}} - \frac{1}{2} \gamma \right]}{\sqrt{\frac{1-e^{-2\gamma}}{2\gamma}}} \\
& = \sqrt{\frac{2\gamma}{1-e^{-2\gamma}}} \times \exp \left[\frac{\alpha^2}{2\gamma^2} \int_0^1 (1-e^{\gamma(s-1)})^2 ds - \left(\int_0^1 [e^{\gamma(s-1)} - e^{2\gamma(s-1)}] ds \right)^2 \frac{\alpha^2/\gamma}{1-e^{-2\gamma}} - \frac{1}{2} \gamma \right] \\
& = \sqrt{\frac{2\gamma e^{-\gamma}}{1-e^{-2\gamma}}} \times \exp \left[\frac{\alpha^2}{2\gamma^2} \times \left(1 - \frac{3-4e^{-\gamma}+e^{-2\gamma}}{2\gamma} \right) - \frac{(1-e^{-\gamma})^4}{4\gamma^2} \times \frac{\alpha^2/\gamma}{1-e^{-2\gamma}} \right] \\
& = \sqrt{\frac{\gamma}{\text{sh } \gamma}} \exp \left[\frac{\alpha^2}{4\gamma^3} (2\gamma - (1-e^{-\gamma})(3-e^{-\gamma})) - \frac{\alpha^2}{4\gamma^3} \frac{(1-e^{-\gamma})^3}{1+e^{-\gamma}} \right] \\
& = \sqrt{\frac{\gamma}{\text{sh } \gamma}} \exp \left[\frac{\alpha^2}{\gamma^3} \left(\frac{\gamma}{2} - \text{th} \left(\frac{\gamma}{2} \right) \right) \right] \cdot \diamond
\end{aligned}$$

Remark 7.3 The above corollary 7.2 contains the following well known particular case : for any real constant γ , we have

$$\mathbb{E}_0^0 \left[\exp \left(-\frac{\gamma^2}{2} \int_0^1 \omega_s^2 ds \right) \right] = \sqrt{\frac{\gamma}{\text{sh } \gamma}}.$$

Now for $0 < \beta < 2$, by (14) we have

$$\mathbb{E}_0^0 \left[\exp \left(\beta \int_0^1 \omega_s^2 ds \right) \right] \leq \mathbb{E}_0^0 \left[e^{\beta (\omega^*)^2} \right] = \beta \int_0^\infty \mathbb{P}_0^0 [\omega^* > \sqrt{t}] e^{\beta t} dt \leq 2\beta \int_0^\infty e^{(\beta-2)t} dt < \infty.$$

By analytical continuation, this implies that

$$\mathbb{E}_0^0 \left[\exp \left(\frac{\gamma^2}{2} \int_0^1 \omega_s^2 ds \right) \right] = \sqrt{\frac{\gamma}{\sin \gamma}}$$

is analytical on $\{0 \leq \gamma < 2\}$, and then on $\{0 \leq \gamma < \pi\}$ as well by monotone convergence (and it explodes at $\gamma = \pi$). Similarly, for any positive q we have :

$$\begin{aligned} \mathbb{E}_0^0 \left[\exp \left(q\beta \int_0^1 \omega + \beta \int_0^1 \omega^2 \right) \right] &\leq \mathbb{E}_0^0 \left[e^{q\beta \omega^* + \beta (\omega^*)^2} \right] = \beta \int_0^\infty \mathbb{P}_0^0 [(\omega^*)^2 + q\omega^* > t] e^{\beta t} dt \\ &= \beta \int_0^\infty \mathbb{P}_0^0 [2\omega^* > \sqrt{q^2 + 4t} - q] e^{\beta t} dt \leq 2\beta \int_0^\infty e^{(\beta-2)t + q\sqrt{q^2 + 4t} - q^2} dt < \infty. \end{aligned}$$

This entails that the function $(\alpha, \beta) \mapsto \mathbb{E}_0^0 \left[\exp \left(\int_0^1 [\alpha \omega_s + \beta \omega_s^2] ds \right) \right]$ is analytical on $\mathbb{C} \times \{\Re(\beta) < 2\}$, and even on $\mathbb{C} \times \{\Re(\beta) < \pi^2/2\}$: $\pi^2/2$ is the abscissa of convergence.

Now the expression found in Corollary 7.2, valid for $-\infty < \beta < 2$ (and not for $\beta = \pi^2/2$), is not straightforwardly analytically continued in the whole half plane $\{\Re(\beta) < 2\}$, whereas we need an explicit analytic continuation. The delicate point is of course the square root coming in the expression of Corollary 7.2. We shall use the following lemma.

Lemma 7.4 (i) For any positive real a, b , we have

$$\text{sh}[a + \sqrt{-1} b] = \sqrt{\frac{\text{ch}(2a) - \cos(2b)}{2}} \times \exp \left[\sqrt{-1} \int_0^b \frac{\text{sh}(2a) d\beta}{\text{ch}(2a) - \cos(2\beta)} \right].$$

(ii) For any real χ and positive x , we have the following continuous lift (i.e., determination) of the square root :

$$\sqrt{\frac{\sqrt{\chi} + \sqrt{-1} x}{\text{sh} \sqrt{\chi} + \sqrt{-1} x}} = \left[\frac{2 \sqrt{\chi^2 + x^2}}{\text{ch} \sqrt{2 \sqrt{\chi^2 + x^2} + 2\chi} - \cos \sqrt{2 \sqrt{\chi^2 + x^2} - 2\chi}} \right]^{1/4} \times e^{\sqrt{-1} \varphi(\chi, x)},$$

with

$$\begin{aligned}
\varphi(\chi, x) &= \frac{1}{2} \operatorname{arctg} \sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{\sqrt{\chi^2 + x^2} + \chi}} - \frac{1}{2} \int_0^{\sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{2}}} \frac{\operatorname{sh} \sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} d\beta}{\operatorname{ch} \sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} - \cos(2\beta)} \\
&= \frac{1}{2} \operatorname{arctg} \sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{\sqrt{\chi^2 + x^2} + \chi}} - \frac{k\pi}{2} - \frac{1}{2} \operatorname{arctg} \left(\operatorname{tg} \left[\sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{2}} - k\pi \right] \times \coth \left[\sqrt{\frac{\sqrt{\chi^2 + x^2} + \chi}{2}} \right] \right) \\
&\text{for } k \in \mathbb{N} \text{ and } \left| \sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{2}} - k\pi \right| \leq \frac{\pi}{2}.
\end{aligned}$$

Proof (i) The modulus of $\operatorname{sh}[a + \sqrt{-1} b] = \operatorname{sh} a \cos b + \sqrt{-1} \operatorname{ch} a \sin b$ is $|\operatorname{sh}[a + \sqrt{-1} b]| = \sqrt{\operatorname{sh}^2 a + \sin^2 b} = \sqrt{\frac{\operatorname{ch}(2a) - \cos(2b)}{2}} > 0$, and setting $\frac{\operatorname{sh}[a + \sqrt{-1} b]}{\sqrt{\operatorname{sh}^2 a + \sin^2 b}} = e^{\sqrt{-1} \varphi_a(b)}$ with $\varphi_a(0) = 0$, we have $\varphi'_a(b) = \frac{\operatorname{sh}(2a)}{\operatorname{ch}(2a) - \cos(2b)}$, since

$$\frac{\operatorname{sh} a \cos b \times \varphi'_a(b)}{\sqrt{\operatorname{sh}^2 a + \sin^2 b}} = \frac{\partial}{\partial b} \sin \varphi_a(b) = \frac{\partial}{\partial b} \frac{\operatorname{ch} a \sin b}{\sqrt{\operatorname{sh}^2 a + \sin^2 b}} = \frac{\operatorname{ch} a \cos b \times \operatorname{sh}^2 a}{(\operatorname{sh}^2 a + \sin^2 b)^{3/2}}.$$

Finally

$$\varphi_a(b) = \int_0^b \frac{\operatorname{sh}(2a) d\beta}{\operatorname{ch}(2a) - \cos(2\beta)}, \text{ whence } \varphi_a(b) = \operatorname{arctg} [\coth a \times \operatorname{tg} b] \text{ for } 0 \leq b \leq \pi/2,$$

and then $\varphi_a(b) = k\pi + \operatorname{arctg} [\coth a \times \operatorname{tg}(b - k\pi)]$ for $k\pi - \frac{\pi}{2} \leq b \leq k\pi + \frac{\pi}{2}$, $k \in \mathbb{N}$.

In particular, we have $\varphi_a(k\pi/2) = k\pi/2$ for any $k \in \mathbb{N}$.

(ii) Taking the usual determination of the square root on $\mathbb{C} \setminus \sqrt{-1} \mathbb{R}_-$, we first have

$$\sqrt{\chi + \sqrt{-1} x} = a + \sqrt{-1} b \text{ with } a := \sqrt{\frac{\sqrt{\chi^2 + x^2} + \chi}{2}}, b := \sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{2}}, \quad (37)$$

so that applying (i) above we have

$$\frac{\sqrt{\chi + \sqrt{-1} x}}{\operatorname{sh} \sqrt{\chi + \sqrt{-1} x}} = \sqrt{\frac{2\sqrt{\chi^2 + x^2}}{\operatorname{ch} \sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} - \cos \sqrt{2\sqrt{\chi^2 + x^2} - 2\chi}}} \times \exp[\sqrt{-1} (\operatorname{arctg}(b/a) - \varphi_a(b))]. \diamond$$

Corollary 7.5 For any real ξ , any $\chi > -\pi^2$ and $x > 0$, we have :

$$\begin{aligned}
\mathbb{E}_0^0 \left[\exp \left(\int_0^1 \left[\sqrt{-1} \xi \omega_s - \frac{\chi + \sqrt{-1} x}{2} \omega_s^2 \right] ds \right) \right] &= e^{\sqrt{-1} \varphi(\chi, x)} \times \\
&\left[\frac{2\sqrt{\chi^2 + x^2}}{\operatorname{ch} \sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} - \cos \sqrt{2\sqrt{\chi^2 + x^2} - 2\chi}} \right]^{1/4} \times \exp \left[\frac{-\xi^2/2}{\chi + \sqrt{-1} x} \times f(\chi, x) \right],
\end{aligned}$$

with $\varphi(\chi, x)$ as in Lemma 7.4, and setting $\sqrt{\chi + \sqrt{-1} x} = a + \sqrt{-1} b$ as in Lemma 7.4 :

$$f(\chi, x) := 1 - \frac{\operatorname{th}[\sqrt{\chi + \sqrt{-1} x}/2]}{\sqrt{\chi + \sqrt{-1} x}/2} = 1 - \frac{a \operatorname{sh} a + b \sin b + \sqrt{-1} (a \sin b - b \operatorname{sh} a)}{(a^2 + b^2)(\operatorname{ch} a + \cos b)/2}. \quad (38)$$

Proof Remark 7.3 and Lemma 7.4 allow to continue Corollary 7.2 analytically, to $\gamma = \sqrt{\chi + \sqrt{-1}x}$, at least for $\chi > -\pi^2$ and $x > 0$. Lemma continues the term $\sqrt{\frac{\gamma}{\text{sh } \gamma}}$ of Corollary 7.2. And the remaining term is

$$\exp\left[\frac{-\xi^2}{2\gamma^2}\left(1 - \frac{2}{\gamma} \times \text{th}\left[\frac{\gamma}{2}\right]\right)\right] = \exp\left[\frac{-\xi^2/2}{\chi + \sqrt{-1}x} f(\chi, x)\right],$$

whence the first expression of the statement for $f(\chi, x)$. Using the notation a, b , the second expression follows at once from the observation that $\text{th}\left[\frac{\sqrt{\chi + \sqrt{-1}x}}{2}\right] = \frac{\text{sh } a + \sqrt{-1} \sin b}{\text{ch } a + \cos b}$. \diamond

8 Evaluation of the oscillatory integral $\mathcal{I}_\varepsilon(y, z)$

Proposition 6.3.1 yields the wanted equivalent for $p_\varepsilon(0; (0, y, z))$, in terms of the stochastic oscillatory integral $\mathcal{I}_\varepsilon(y, z)$, which we have to evaluate carefully. We shall first use Section 7, in order to express $\mathcal{I}_\varepsilon(y, z)$ as a finite-dimensional oscillatory integral. Even so, it will however remain delicate to handle.

8.1 Reduction to a finite-dimensional oscillatory integral

First we have the following.

Lemma 8.1.1 *For any $(y, z) \in \mathbb{R}^2$ and $\varepsilon > 0$ we have*

$$\begin{aligned} \mathcal{I}_\varepsilon(y, z) &= 2 \Re \left\{ \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right) \tilde{\mathcal{E}}_0(x, \xi) dx \right] d\xi \right\} \\ &= 2 \Re \left\{ \int_0^\infty \exp\left(\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right) \left[\int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \tilde{\mathcal{E}}_0(x, \xi) d\xi \right] dx \right\}. \end{aligned}$$

Proof Note first the obvious symmetries: $\tilde{\mathcal{E}}_0(\xi', \xi) = \tilde{\mathcal{E}}_0(\xi', -\xi) = \overline{\tilde{\mathcal{E}}_0(-\xi', \xi)}$, due to the symmetry of \mathbb{P}_0^0 , which permit in particular to restrict below to $\xi' \geq 0$. Note that $\tilde{\mathcal{E}}_0$ is a continuous function on \mathbb{R}^2 , such that $\tilde{\mathcal{E}}_0(0, \xi) = e^{-\xi^2/24}$ by Lemma 5.2. Since $\tilde{\mathcal{E}}_0(\xi', \xi) \in L^1(\mathbb{R}^2, d\xi' d\xi)$, we can apply Fubini's Theorem, performing the integration with respect to ξ' first. Using both symmetries of $\tilde{\mathcal{E}}_0(\cdot, \xi)$, this yields:

$$\begin{aligned} \mathcal{I}_\varepsilon(y, z) &= \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_{-\infty}^\infty \exp\left(\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} \xi'\right) \tilde{\mathcal{E}}_0(\xi', \xi) d\xi' \right] d\xi \\ &= \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} \xi'\right) \tilde{\mathcal{E}}_0(\xi', \xi) d\xi' \right] d\xi \\ &\quad + \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(-\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} \xi'\right) \tilde{\mathcal{E}}_0(-\xi', \xi) d\xi' \right] d\xi \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(\frac{1-y}{\varepsilon} \sqrt{-1} \xi'\right) \tilde{\mathcal{E}}_0(\xi', \xi) d\xi' \right] d\xi \\
&\quad + \int_{\mathbb{R}} \exp\left[-\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(-\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} \xi'\right) \overline{\tilde{\mathcal{E}}_0(\xi', \xi)} d\xi' \right] d\xi \\
&= 2 \Re \left\{ \int_{\mathbb{R}} \exp\left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi\right] \times \left[\int_0^\infty \exp\left(\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} \xi'\right) \tilde{\mathcal{E}}_0(\xi', \xi) d\xi' \right] d\xi \right\}.
\end{aligned}$$

Finally we can apply Fubini's Theorem again, integrating now with respect to ξ first. \diamond

Now we use Section 7, to deduce the following expression for the integral $\mathcal{I}_\varepsilon(y, z)$ (we have to evaluate), as a finite-dimensional oscillatory integral.

Proposition 8.1.2 *For any $(y, z) \in \mathbb{R}^2$ and $\varepsilon > 0$ we have*

$$\mathcal{I}_\varepsilon(y, z) = 2 \Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x - \frac{z^2}{2\varepsilon^3} \frac{\sqrt{-1} x}{f(0, x)} \right] \Phi(x) dx \right\},$$

where Φ is the analytical function given (for any $x \geq 0$) by :

$$\Phi(x) := e^{\sqrt{-1} \varphi(x)} \times \left[\frac{2x}{\operatorname{ch}\sqrt{2x} - \cos\sqrt{2x}} \right]^{1/4} \times \sqrt{\frac{2\pi\sqrt{-1}x}{f(x)}},$$

$$\frac{\sqrt{-1}x}{f(0, x)} = \frac{x\sqrt{x/2} (\operatorname{ch}\sqrt{x/2} + \cos\sqrt{x/2})}{\operatorname{sh}\sqrt{x/2} - \sin\sqrt{x/2} + \sqrt{-1} (\operatorname{sh}\sqrt{x/2} + \sin\sqrt{x/2} - (\operatorname{ch}\sqrt{x/2} + \cos\sqrt{x/2})\sqrt{x/2})},$$

and

$$\varphi(0) = 0, \quad \varphi'(x) = \frac{-1}{2\sqrt{2x}} \times \frac{\operatorname{sh}\sqrt{2x} - \sin\sqrt{2x}}{\operatorname{ch}\sqrt{2x} - \cos\sqrt{2x}}.$$

Proof By Corollary 7.5, for any real $\chi > -\pi^2$ and $x \geq 0$ we have

$$\begin{aligned}
\tilde{\mathcal{E}}_0(x - \sqrt{-1}\chi, \xi) &= \mathbb{E}_0^0 \left[\exp \left(\int_0^1 \left[\sqrt{-1}\xi \omega_s - \frac{\chi + \sqrt{-1}x}{2} \omega_s^2 \right] ds \right) \right] \quad (39) \\
&= e^{\sqrt{-1} \varphi(\chi, x)} \times \left[\frac{2\sqrt{\chi^2 + x^2}}{\operatorname{ch}\sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} - \cos\sqrt{2\sqrt{\chi^2 + x^2} - 2\chi}} \right]^{1/4} \times \exp \left[\frac{-\xi^2/2}{\chi + \sqrt{-1}x} \times f(\chi, x) \right],
\end{aligned}$$

with $f(\chi, x)$ given by (38). In particular,

$$\tilde{\mathcal{E}}_0(x, \xi) = e^{\sqrt{-1} \varphi(x)} \times \left[\frac{2x}{\operatorname{ch}\sqrt{2x} - \cos\sqrt{2x}} \right]^{1/4} \times \exp \left[\frac{-\xi^2 f(0, x)}{2\sqrt{-1}x} \right],$$

with

$$\frac{f(0, x)}{\sqrt{-1}} = \frac{\operatorname{sh}\sqrt{x/2} - \sin\sqrt{x/2} + \sqrt{-1} (\operatorname{sh}\sqrt{x/2} + \sin\sqrt{x/2})}{(\operatorname{ch}\sqrt{x/2} + \cos\sqrt{x/2})\sqrt{x/2}} - \sqrt{-1},$$

and by Lemma 7.4: $\varphi'(x) = \frac{-1}{2\sqrt{2x}} \times \frac{\text{sh}\sqrt{2x} - \sin\sqrt{2x}}{\text{ch}\sqrt{2x} - \cos\sqrt{2x}},$ and

$$\varphi(x) = \frac{\pi}{8} - \frac{k\pi}{2} - \frac{1}{2} \arctg \left[\text{tg} \left(\sqrt{\frac{x}{2}} - k\pi \right) \times \coth \sqrt{\frac{x}{2}} \right] \quad \text{for } k \in \mathbb{N} \text{ and } \left| \sqrt{\frac{x}{2}} - k\pi \right| \leq \frac{\pi}{2}.$$

Hence we obtain :

$$\int_{\mathbb{R}} \exp \left[\sqrt{-1} \frac{z}{\varepsilon^{3/2}} \xi \right] \times \tilde{\mathcal{E}}_0(x, \xi) d\xi = e^{\sqrt{-1}\varphi(x)} \left[\frac{2x}{\text{ch}\sqrt{2x} - \cos\sqrt{2x}} \right]^{1/4} \sqrt{\frac{2\pi\sqrt{-1}x}{f(0,x)}} \times \exp \left[-\frac{z^2}{2\varepsilon^3} \times \frac{\sqrt{-1}x}{f(0,x)} \right].$$

Therefore, using Lemma 8.1.1 we obtain the expression of the statement.

Note then that $\text{ch}\sqrt{2x} - \cos\sqrt{2x} = 0 \Leftrightarrow \sqrt{2x} \in (\sqrt{-1} \pm 1)\pi\mathbb{Z} \Leftrightarrow x \in \pm\sqrt{-1}\pi^2\mathbb{N}^2.$

Note also that $\frac{\sqrt{-1}x}{f(0,x)}$ is an even meromorphic function of \sqrt{x} , locally expandable in a power series of x . Moreover, we have

$$\frac{\sqrt{-1}}{f(0, 2x^2)} = \frac{\sqrt{-1} \left(1 - \frac{\text{sh } x + \sin x}{x(\text{ch } x + \cos x)} \right) + \frac{\text{sh } x - \sin x}{x(\text{ch } x + \cos x)}}{\left(1 - \frac{\text{sh } x + \sin x}{x(\text{ch } x + \cos x)} \right)^2 + \left(\frac{\text{sh } x - \sin x}{x(\text{ch } x + \cos x)} \right)^2}. \quad (40)$$

Since $\left(x - \frac{\text{sh } x + \sin x}{\text{ch } x + \cos x} \right)$ has derivative $\frac{\text{sh } 2x - \sin^2 x}{(\text{ch } x + \cos x)^2} > 0$ on \mathbb{R}_+^* and then continuously increases from 0 to infinity, we have $\left(1 - \frac{\text{sh } x + \sin x}{x(\text{ch } x + \cos x)} \right) > 0$ on \mathbb{R}_+^* , as well as $x \frac{\text{sh } x - \sin x}{\text{ch } x + \cos x} > 0$.

In particular the real and imaginary parts in (40) are positive continuous on \mathbb{R}_+^* , which shows that in the above the argument of $\frac{\sqrt{-1}x}{f(0,x)}$ belongs to $[0, \pi/2]$, confirming that in the expression for $\Phi(x)$ we could indeed use the usual continuous determination of the square root. Finally the expression of Φ shows indeed that it is analytical on \mathbb{R}_+ , as the product of three analytical terms. \diamond

To handle the oscillatory integral yielding $\mathcal{I}_\varepsilon(y, z)$ in Proposition 8.1.2, as is usual we shall change the path of integration. However, for that we need an analytic continuation, which demands more work. Part of it was already done in Section 7. The following is another step in this direction.

Lemma 8.1.3 *The function $\frac{\sqrt{-1}x}{f(0,x)}$ is meromorphic on $\mathbb{C} \setminus \mathbb{R}_-$, and its poles are the values $4\sqrt{-1}\theta_k^2$, $k \geq 1$, where $\{\theta_1 < \theta_2 < \dots\}$ are the positive roots of $\text{tg } \theta_k = \theta_k$. In particular, $\frac{4\pi}{3} < \theta_1 < \frac{3\pi}{2}$. We have $f(0, x - \sqrt{-1}\chi) = f(\chi, x)$. Moreover $\frac{\sqrt{-1}x}{f(0,x)}$ is real on the segment $[0, 4\sqrt{-1}\theta_1^2[$, positive on $]0, \sqrt{-1}\pi^2[$ and negative on $] \sqrt{-1}\pi^2, 4\sqrt{-1}\theta_1^2[$.*

Proof By (38), we have $f(0, x) = 1 - \frac{\text{th}[\sqrt{\sqrt{-1}x}/2]}{\sqrt{\sqrt{-1}x}/2}$, hence $f(0, -4\sqrt{-1}x^2) = 1 - \frac{\text{th } x}{x}$.

Now the equation $re^{\sqrt{-1}\theta} = \text{th}(re^{\sqrt{-1}\theta})$ has no solution for $r > 0$ and $|\theta| < \pi/2$.

Indeed, this equation is equivalent to both $r = \cos \theta \operatorname{th}(r \cos \theta) + \sin \theta \operatorname{tg}(r \sin \theta)$ and $r \operatorname{th}(r \cos \theta) \operatorname{tg}(r \sin \theta) = \cos \theta \operatorname{tg}(r \sin \theta) - \sin \theta \operatorname{th}(r \cos \theta)$, which by eliminating r implies $\operatorname{tg} \theta = \frac{\sin(2r \sin \theta)}{\operatorname{sh}(2r \cos \theta)} < \operatorname{tg} \theta$ for $0 < \theta < \pi/2$, a contradiction. By symmetry, the claim holds for negative θ as well, and also clearly for $\theta = 0$. On the contrary, this does not hold for $\theta = \pi/2$, since the imaginary roots are $\pm \sqrt{-1} \theta_k$. Thus $\frac{\sqrt{-1}(-4\sqrt{-1}x^2)}{f(0, -4\sqrt{-1}x^2)} = \frac{4x^3}{x - \operatorname{th} x}$ is even holomorphic in $\{|\arg(x)| < \pi/2\}$, whereas $\frac{\sqrt{-1}(4\sqrt{-1}x^2)}{f(0, 4\sqrt{-1}x^2)} = \frac{4x^3}{\operatorname{tg} x - x}$ has poles.

By symmetry we can change θ into $\theta + \pi$, which entails the analyticity for $\pi > \arg(x) > \pi/2$ as well. Finally, from the expression in Proposition 8.1.2 we get

$$\frac{\sqrt{-1}(4\sqrt{-1}x^2)}{f(0, 4\sqrt{-1}x^2)} = \frac{4x^3 \cos x}{\sin x - x \cos x} \cdot \diamond$$

Remark 8.1.4 The dominant part of the integral in Proposition 8.1.2 happens to belong to $\sqrt{-1} \mathbb{R}$, and then does not contribute to the real projection. This means that the contribution to the heat kernel we have to extract is only a very small part of this integral. Therefore handling it remains delicate. We shall proceed by first changing the contour, in order to suppress the dominant hanger-on contribution. A good contour must go through a convenient saddle point of the considered oscillatory integral. However in the present case the saddle points are not easy to find.

8.2 First sub-case: $y \leq 0, z = 0$

We deal here with the sub-case $y \leq 0$, in (recall Proposition 8.1.2)

$$\mathcal{I}_\varepsilon(y, 0) = 2 \Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(x) dx \right\}.$$

The dominant saddle point of the above oscillatory integral, happens to be close to $4\sqrt{-1} \left(\pi^2 - \frac{\varepsilon^2}{4(\varepsilon - y)} \right)$, as will be confirmed later, by Proposition 8.2.10. The point $\sqrt{-1} \pi^2$ (though a root of $\operatorname{ch} \sqrt{2x} - \cos \sqrt{2x}$ too, as mentioned in the proof of Proposition 8.1.2) is actually not a true pole, as will be clear in Lemma 8.2.2 below.

Thus we need to change the path of integration and to use an analytical continuation of the function Φ on the closure of a convenient domain. We already have part of this continuation, owing to Lemma 7.4, Corollary 7.5 and (39)(38).

For the new contour, we need to change x into $x - \sqrt{-1} \chi$, with $x \geq 0 \geq \chi \geq 4(\eta - \pi^2)$ and $\eta = \frac{\varepsilon^2}{4(\varepsilon - y)} > 0$. According to (39) and Proposition 8.1.2, the function Φ admits the following continuation at least in $] - \pi^2, \infty[\times]0, \infty[$:

$$\Phi(x - \sqrt{-1} \chi) \equiv \Phi(\chi, x) := e^{\sqrt{-1} \varphi(\chi, x)} \times \psi(\chi, x) \times \sqrt{2\pi \frac{\chi + \sqrt{-1} x}{f(\chi, x)}}, \quad (41)$$

where

$$\begin{aligned}\psi(\chi, x) &:= \left[\frac{2\sqrt{\chi^2 + x^2}}{\operatorname{ch}\sqrt{2\sqrt{\chi^2 + x^2} + 2\chi} - \cos\sqrt{2\sqrt{\chi^2 + x^2} - 2\chi}} \right]^{1/4} \\ &= \left[\frac{a^2 + b^2}{(\operatorname{ch} a - \cos b)(\operatorname{ch} a + \cos b)} \right]^{1/4} \quad \text{with } a, b \text{ as in (37) and Lemma 8.2.1 below,}\end{aligned}\tag{42}$$

$\varphi(\chi, x)$ is given by Lemma 7.4, $f(\chi, x)$ is given by (38) (in Corollary 7.5). Now, in order that the above continuation be analytic, we need a continuous lift of the square root

$\sqrt{\frac{\chi + \sqrt{-1}x}{f(\chi, x)}}$, for $-\pi^2 - \eta \leq \chi \leq 0 \leq x$. It is provided as follows.

Lemma 8.2.1 *The square root $\sqrt{\frac{\chi + \sqrt{-1}x}{f(\chi, x)}}$ admits the following analytical lift, for any χ, x such that $\chi > -4\theta_1^2$ and $x > 0$:*

$$\sqrt{\frac{\chi + \sqrt{-1}x}{f(\chi, x)}} = \frac{(a^2 + b^2)(\operatorname{ch} a + \cos b)^{1/4} \times e^{\sqrt{-1}\tilde{\varphi}(\chi, x)/2}}{\left[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b) \right]^{1/4}}$$

with the notation (37):

$$\sqrt{\chi + \sqrt{-1}x} = a + \sqrt{-1}b \quad \text{where} \quad a := \sqrt{\frac{\sqrt{\chi^2 + x^2} + \chi}{2}}, \quad b := \sqrt{\frac{\sqrt{\chi^2 + x^2} - \chi}{2}},$$

and the (bounded for bounded χ) argument:

$$\begin{aligned}\tilde{\varphi}(\chi, x) &= \arg\left(\frac{\sqrt{-1}x}{f(0, x)}\right) + \frac{\pi}{2} - \frac{3}{2} \operatorname{arctg}\left[\frac{\chi}{x}\right] + \\ &+ \int_0^x \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2 + x^2}(\operatorname{ch} a + \cos b)[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)]} d\chi.\end{aligned}$$

Proof According to (38), we have

$$\frac{1}{f(\chi, x)} = \frac{(a^2 + b^2)(\operatorname{ch} a + \cos b)}{(a^2 + b^2)(\operatorname{ch} a + \cos b) - 2a \operatorname{sh} a - 2b \sin b + 2\sqrt{-1}(b \operatorname{sh} a - a \sin b)} = \varrho e^{\sqrt{-1}\alpha}$$

(which is real for $x = 0$), with $\alpha \in \mathbb{R}$,

$$\varrho := \frac{(a^2 + b^2)(\operatorname{ch} a + \cos b)}{\sqrt{[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 2a \operatorname{sh} a - 2b \sin b]^2 + 4(b \operatorname{sh} a - a \sin b)^2}} = \frac{\sqrt{a^2 + b^2}(\operatorname{ch} a + \cos b)}{\sqrt{B}},$$

$$\begin{aligned}B &:= (a^2 + b^2)(\operatorname{ch} a + \cos b)^2 - 4(a \operatorname{sh} a + b \sin b)(\operatorname{ch} a + \cos b) + 4(\operatorname{sh}^2 a + \sin^2 b) \\ &= (\operatorname{ch} a + \cos b) \times [(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)],\end{aligned}$$

and

$$\cos \alpha = \frac{(a^2 + b^2)^{1/2} (\operatorname{ch} a + \cos b) - 2(a^2 + b^2)^{-1/2} (a \operatorname{sh} a + b \sin b)}{\sqrt{B}},$$

$$\sin \alpha = \frac{2(a^2 + b^2)^{-1/2} (a \sin b - b \operatorname{sh} a)}{\sqrt{B}}.$$

Note that by Lemma 8.1.3, $B \neq 0$ for $\chi > -4\theta_1^2$, hence for $\chi > -7\pi^2$.

Thence, we successively have :

$$\frac{\partial B}{\partial a} = 2(\operatorname{ch} a + \cos b) [a(\cos b - \operatorname{ch} a) + (a^2 + b^2 - 2) \operatorname{sh} a] - 4(a \operatorname{sh} a + b \sin b - 2 \operatorname{ch} a) \operatorname{sh} a;$$

$$\frac{\partial B}{\partial b} = 2(\operatorname{ch} a + \cos b) [b(\operatorname{ch} a - \cos b) - (a^2 + b^2 + 2) \sin b] + 4(a \operatorname{sh} a + b \sin b + 2 \cos b) \sin b;$$

$$\frac{1}{2} \left[a \frac{\partial B}{\partial a} - b \frac{\partial B}{\partial b} \right] = [a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b] [(a^2 + b^2)(\operatorname{ch} a + \cos b) - 2(a \operatorname{sh} a + b \sin b)];$$

$$\frac{1}{2} \cos \alpha \frac{\partial \alpha}{\partial a} = \frac{\partial \sin \alpha}{2 \partial a} = \frac{\partial}{\partial a} \frac{(a^2 + b^2)^{-1/2} (a \sin b - b \operatorname{sh} a)}{\sqrt{B}} = \frac{A}{B^{3/2} (a^2 + b^2)^{3/2}};$$

$$\frac{1}{2} \cos \alpha \frac{\partial \alpha}{\partial b} = \frac{\partial \sin \alpha}{2 \partial b} = \frac{\partial}{\partial b} \frac{(a^2 + b^2)^{-1/2} (a \sin b - b \operatorname{sh} a)}{\sqrt{B}} = \frac{A'}{B^{3/2} (a^2 + b^2)^{3/2}};$$

with

$$A := (a \operatorname{sh} a + b \sin b - (a^2 + b^2) \operatorname{ch} a) b B + \frac{1}{2} (a^2 + b^2) (b \operatorname{sh} a - a \sin b) \frac{\partial B}{\partial a};$$

$$A' := ((a^2 + b^2) \cos b - a \operatorname{sh} a - b \sin b) a B + \frac{1}{2} (a^2 + b^2) (b \operatorname{sh} a - a \sin b) \frac{\partial B}{\partial b};$$

$$\begin{aligned} a A - b A' &= [2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)] ab B \\ &\quad + \frac{1}{2} (a^2 + b^2) (b \operatorname{sh} a - a \sin b) \left[a \frac{\partial B}{\partial a} - b \frac{\partial B}{\partial b} \right] \\ &= [(a^2 + b^2)(\operatorname{ch} a + \cos b) - 2(a \operatorname{sh} a + b \sin b)] \times \\ &\quad \times ((a^2 + b^2) (b \operatorname{sh} a - a \sin b) [a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b] - ab B). \end{aligned}$$

Hence

$$\frac{\partial}{\partial \chi} = \frac{\partial b}{\partial \chi} \frac{\partial}{\partial b} + \frac{\partial a}{\partial \chi} \frac{\partial}{\partial a} = \frac{1}{2(a^2 + b^2)} \left(a \frac{\partial}{\partial a} - b \frac{\partial}{\partial b} \right); \quad \cos \alpha \frac{\partial \alpha}{\partial \chi} = \frac{a A - b A'}{B^{3/2} (a^2 + b^2)^{5/2}};$$

$$\frac{\partial \alpha}{\partial \chi} = \frac{(b \operatorname{sh} a - a \sin b) [a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b]}{\sqrt{\chi^2 + x^2} B} - \frac{x}{2(\chi^2 + x^2)},$$

since $(a^2 + b^2) = \sqrt{\chi^2 + x^2}$ and $x = 2ab$. Then

$$\frac{\chi + \sqrt{-1} x}{f(\chi, x)} = (\chi + \sqrt{-1} x) \varrho e^{\sqrt{-1} \alpha} = (a^2 + b^2) \varrho e^{\sqrt{-1} \varphi(\chi, x)}$$

with $\tilde{\varphi}(\chi, x) = \alpha + \pi 1_{\{\chi \leq 0\}} + \operatorname{arctg}(x/\chi) = \alpha + \frac{\pi}{2} - \int_0^x \frac{x d\chi}{\chi^2 + x^2}$.

Finally we obtain

$$\begin{aligned} \tilde{\varphi}(\chi, x) &= \frac{\pi}{2} - \int_0^x \frac{x d\chi}{\chi^2 + x^2} + \arg \left[\frac{\sqrt{-1} x}{f(0, x)} \right] + \int_0^x \frac{\partial \alpha}{\partial \chi} d\chi \\ &= \frac{\pi}{2} - \frac{3}{2} \operatorname{arctg} \left[\frac{\chi}{x} \right] + \arg \left[\frac{\sqrt{-1} x}{f(0, x)} \right] + \int_0^x \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2 + x^2} B} d\chi. \end{aligned}$$

We already noticed that $0 \leq \arg \left[\frac{\sqrt{-1} x}{f(0, x)} \right] \leq \frac{\pi}{2}$. Moreover, as a, b grow with x , asymptotically as \sqrt{x} , the above integral is bounded too, provided χ remains bounded. \diamond

Lemma 8.2.1, (41) and the expression following (42) (i.e., ψ in terms of a, b) together easily yield the following.

Lemma 8.2.2 *For any real χ, x such that $x > 0$ we have the following analytical lift:*

$$\Phi(\chi, x) = \frac{(a^2 + b^2)^{5/4} (\operatorname{ch} a - \cos b)^{-1/4} \times e^{\sqrt{-1}(\varphi(\chi, x) + \tilde{\varphi}(\chi, x)/2)}}{\left[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b) \right]^{1/4}},$$

with the notation (37) as in Lemma 8.2.1, $\varphi(\chi, x)$ given by Lemma 7.4, and the same argument $\tilde{\varphi}(\chi, x)$ as in Lemma 8.2.1.

Moreover the above extends continuously to $x = 0$, except for the singular points where the denominator vanishes, i.e., $\chi = -4\theta_k^2$ ($k \geq 1$, recall Lemma 8.1.3), and $\chi = -4k^2\pi^2$ ($k \geq 1$, namely the zeros of $(\operatorname{ch} a - \cos b)$).

Let us consider the contour

$$\Gamma_\eta := \left[0, 4\sqrt{-1}(\pi^2 - \eta) \right] \cup \left(4\sqrt{-1}(\pi^2 - \eta) + \mathbb{R}_+ \right). \quad (43)$$

Lemma 8.2.2 allows to change the contour in $\Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(x) dx \right\}$, into Γ_η of (43), in order to obtain the first reduction announced in Remark 8.1.4, namely the following.

Proposition 8.2.3 *For any $y \leq 0$ and small $\varepsilon, \eta > 0$ we have*

$$\mathcal{I}_\varepsilon(y, 0) = \exp \left[\frac{y - \varepsilon}{\varepsilon^2} 4(\pi^2 - \eta) \right] \times 2 \Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(4\eta - 4\pi^2, x) dx \right\},$$

where $\Phi(\chi, x) \equiv \Phi(x - \sqrt{-1}\chi)$ was defined in (41).

Proof The above-mentioned change of contour (43) is justified by

$$\limsup_{R \rightarrow \infty} \left| \int_0^{4\pi^2 - \eta} \exp \left[\frac{y - \varepsilon}{\varepsilon^2} (x - \sqrt{-1} R) \right] \Phi(-x, R) dx \right| \leq \limsup_{R \rightarrow \infty} \int_{4\eta - 4\pi^2}^0 |\Phi(\chi, R)| d\chi,$$

as Lemma 8.2.2 ensures that the latter vanishes: actually it shows that $|\Phi(\chi, R)| = \mathcal{O}(R e^{-\sqrt{R/8}})$, uniformly for bounded χ . Thus performing this change yields:

$$\begin{aligned} \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(x) dx &= \int_{\Gamma_\eta} \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(x) dx \\ &= \sqrt{-1} \int_0^{4\pi^2 - \eta} \exp \left[\frac{y - \varepsilon}{\varepsilon^2} x \right] \Phi(-x, 0) dx \\ &\quad + \exp \left[\frac{y - \varepsilon}{\varepsilon^2} 4(\pi^2 - \eta) \right] \times \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(4\eta - 4\pi^2, x) dx. \end{aligned}$$

Then by Lemma 7.4, for $\chi < 0$, letting $x \searrow 0$, we find:

- * $\varphi(\chi, 0) = 0$ separately for $b = \sqrt{-\chi} \in [0, \pi/2]$ and for $b = \sqrt{-\chi} \in]\pi/2, \pi[$;
- * $\varphi(\chi, 0) = -\frac{\pi}{2}$ separately for $b = \sqrt{-\chi} \in]\pi, 3\pi/2[$ and for $b = \sqrt{-\chi} \in]3\pi/2, 2\pi[$.

On the other hand, we saw in Lemma 8.1.3 that $\frac{-x}{f(-x, 0)} \in \mathbb{R}_+$ for $0 \leq x \leq \pi^2$ and $\frac{-x}{f(-x, 0)} \in \mathbb{R}_-$ for $\pi^2 \leq x < 4\theta_1^2$. Hence, according to (41), for all $x \in [0, 4\theta_1^2[$ we have

$$\Phi(-x, 0) = e^{\sqrt{-1} \varphi(-x, 0)} \psi(-x, 0) \sqrt{\frac{-2\pi x}{f(-x, 0)}} \in \mathbb{R},$$

so that

$$\sqrt{-1} \int_0^{4\pi^2 - \eta} \exp \left[\frac{y - \varepsilon}{\varepsilon^2} x \right] \Phi(-x, 0) dx \in \sqrt{-1} \mathbb{R}$$

and then disappears when taking the real part. \diamond

We have now to carefully analyze the integral appearing in Proposition 8.2.3, and then first to approach the term $\Phi(4\eta - 4\pi^2, x)$ near the singularity $4\sqrt{-1}\pi^2$. This is the step where the choice of $4\eta - 4\pi^2$ begins to prove to be the right one. We take $\eta = \frac{\varepsilon^2}{4(\varepsilon - y)}$.

Lemma 8.2.4 *For $y \leq 0$, $r' := \frac{3}{2} - \frac{3}{4} 1_{\{y=0\}} \leq r \leq \frac{9}{10}(2 - 1_{\{y=0\}})$ and $0 \leq x \leq \varepsilon^{r-2}(\varepsilon - y)$, with real $\mathcal{O}()$ and uniformly with respect to $y \leq 0$, we have*

$$\Phi \left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, \frac{\varepsilon^2}{\varepsilon - y} x \right) = \frac{(8\pi^{5/2} + \mathcal{O}(\frac{\varepsilon^2}{\varepsilon - y})) \sqrt{\varepsilon - y}}{(x^2 + 1)^{1/4} \varepsilon} \times e^{\sqrt{-1} (-\frac{3\pi}{8} - \frac{1}{2} \arctg x + \mathcal{O}(\varepsilon^{r/3}))}.$$

Proof Using the notation (37): $\sqrt{\sqrt{-1} x - 4(\pi^2 - \eta)} = a + \sqrt{-1} b$, for small $x > \eta$ we successively have:

$$\sqrt{\chi^2 + x^2} = a^2 + b^2 = 4\pi^2 - 4\eta + \frac{x^2}{8\pi^2} + \mathcal{O}(\eta x^2); \quad a = \sqrt{\frac{x^2}{16\pi^2} + \mathcal{O}(\eta x^2)} = \frac{x}{4\pi} + \mathcal{O}(\eta x);$$

$$b = \sqrt{4\pi^2 - 4\eta + \frac{x^2}{16\pi^2} + \mathcal{O}(\eta x^2)} = 2\pi - \frac{\eta}{\pi} + \frac{x^2 - 16\eta^2}{64\pi^3} + \mathcal{O}(\eta x^2);$$

$$\operatorname{sh} a = \frac{x}{4\pi} + \mathcal{O}(\eta x) = \operatorname{th} a; \quad \operatorname{ch} a = 1 + \frac{x^2}{32\pi^2} + \mathcal{O}(\eta x^2);$$

$$\sin b = -\frac{\eta}{\pi} + \frac{x^2 - 16\eta^2}{64\pi^3} + \mathcal{O}(\eta x^2); \quad \cos b = 1 - \frac{\eta^2}{2\pi^2} + \mathcal{O}(\eta x^2);$$

whence

$$\operatorname{ch} a + \cos b = 2 + \mathcal{O}(x^2); \quad \operatorname{ch} a - \cos b = \frac{x^2 + 16\eta^2}{32\pi^2} + \mathcal{O}(\eta x^2);$$

$$a \operatorname{sh} a + b \sin b = -2\eta + \frac{9x^2 + 16\eta^2}{32\pi^2} + \mathcal{O}(\eta x^2).$$

Therefore, according to Lemma 8.2.2 we obtain:

$$\begin{aligned} \Phi(\chi, x) &= \frac{(4\pi^2 + \mathcal{O}(\eta + x^2))^{5/4} \sqrt{8\pi} \times e^{\sqrt{-1}(\varphi(\chi, x) + \tilde{\varphi}(\chi, x)/2)}}{[16\pi^2 + 8\eta + \mathcal{O}(x^2)]^{1/4} \times (x^2 + 16\eta^2)^{1/4}} \\ &= \frac{8\pi^{5/2} + \mathcal{O}(\eta + x^2)}{(x^2 + 16\eta^2)^{1/4}} \times e^{\sqrt{-1}(\varphi(4\eta - 4\pi^2, x) + \tilde{\varphi}(4\eta - 4\pi^2, x)/2)} \end{aligned}$$

on the one hand, and on the other hand:

$$\tilde{\varphi}(4\eta - 4\pi^2, x) = \frac{\pi}{4} - \frac{3x}{8\pi^2} + \mathcal{O}(x^2) + \int_0^{4\eta - 4\pi^2} \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2 + x^2} B} d\chi,$$

and owing to Lemma 7.4:

$$\begin{aligned} \varphi(4\eta - 4\pi^2, x) &= \frac{1}{2} \operatorname{arctg}(b/a) - \pi - \frac{1}{2} \operatorname{arctg}[\operatorname{tg}(b - 2\pi) \times \operatorname{coth} a] \\ &= -\frac{1}{2} \operatorname{arctg}(a/b) - \pi + \frac{1}{2} \operatorname{arctg}[\operatorname{cotg}(b - 2\pi) \times \operatorname{th} a] \\ &= -\pi - \frac{x}{16\pi^2} + \mathcal{O}(\eta x) - \frac{1}{2} \operatorname{arctg}\left(\frac{x + \mathcal{O}(\eta x)}{4\eta + \frac{16\eta^2 - x^2}{16\pi^2} + \mathcal{O}(\eta x^2)}\right). \end{aligned} \quad (44)$$

Then with $\eta = \frac{\varepsilon^2}{4(\varepsilon - y)}$, for $\frac{3}{4}(2 - 1_{\{y=0\}}) \leq r \leq \frac{9}{10}(2 - 1_{\{y=0\}})$ we have $\varepsilon^{r-2}(\varepsilon - y) \rightarrow \infty$ as $\varepsilon \searrow 0$, and $\varepsilon^{2r} \leq \eta^{3/2}$. Thus for $0 \leq x \leq \varepsilon^{r-2}(\varepsilon - y)$, by the above we have:

$$\begin{aligned} \Phi\left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, \frac{\varepsilon^2}{\varepsilon - y} x\right) &= \frac{\sqrt{\varepsilon - y}}{\varepsilon} \times \frac{8\pi^{5/2} + \mathcal{O}(\frac{\varepsilon^2}{\varepsilon - y})}{(x^2 + 1)^{1/4}} \\ &\times \exp\left[\sqrt{-1}\left(\varphi\left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, \frac{\varepsilon^2}{\varepsilon - y} x\right) + \frac{1}{2} \tilde{\varphi}\left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, \frac{\varepsilon^2}{\varepsilon - y} x\right)\right)\right], \end{aligned}$$

with by (44) (using $\eta \geq \varepsilon^{4r/3}$, hence $x^2\eta \leq \varepsilon^{2r}/\eta \leq \varepsilon^{2r/3}$):

$$\begin{aligned}
\varphi\left(\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y}x\right) &= \mathcal{O}(\varepsilon^r) - \pi - \frac{1}{2} \operatorname{arctg}\left[\frac{x[1+\mathcal{O}(\eta)]}{1+\mathcal{O}(x^2\eta)}\right] \\
&= \mathcal{O}(\varepsilon^r) - \pi - \frac{1}{2} \operatorname{arctg}(x[1+(1+x^2)\mathcal{O}(\eta)]) = \mathcal{O}(\varepsilon^r) - \pi - \frac{1}{2} \operatorname{arctg}(x[1+\mathcal{O}(\varepsilon^{2r/3})]) \\
&= \mathcal{O}(\varepsilon^r) - \pi - \frac{1}{2} \operatorname{arctg} x + \frac{1}{2} \operatorname{arctg}\left(\frac{x\mathcal{O}(\varepsilon^{2r/3})}{1+x^2\mathcal{O}(\varepsilon^{2r/3})}\right) = -\pi - \frac{1}{2} \operatorname{arctg} x + \mathcal{O}(\varepsilon^{r/3}).
\end{aligned}$$

Moreover, by (40) near 0 we have :

$$\frac{\sqrt{-1}x}{f(0,x)} = 6 + \sqrt{-1} \frac{6}{5}x - \frac{x^2}{140} + \mathcal{O}(x^3), \quad (45)$$

and then by Lemma 8.2.1, again for $0 \leq x \leq \varepsilon^{r-2}(\varepsilon-y)$:

$$\begin{aligned}
&\tilde{\varphi}\left(\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y}x\right) \\
&= \arg\left[\frac{\sqrt{-1}x}{f(0,x)}\right] + \frac{5\pi}{4} - \frac{3}{2} \operatorname{arctg}\left[\frac{\mathcal{O}(\varepsilon^r)}{4\pi^2}\right] + \int_0^{\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2} \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2+x^2} B} d\chi \\
&= \frac{5\pi}{4} + \mathcal{O}(\varepsilon^r) + \mathcal{O}(1) \int_0^{\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2} \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2+x^2}} d\chi, \\
&\quad \text{with (for } 0 \geq \chi > -4\pi^2) \quad a = \sqrt{\frac{\sqrt{\chi^2+\mathcal{O}(\varepsilon^{2r})}+\chi}{2}}, \quad b = \sqrt{\frac{\sqrt{\chi^2+\mathcal{O}(\varepsilon^{2r})}-\chi}{2}}.
\end{aligned}$$

Let us distinguish in the above integral between $4\pi^2 > |\chi| \geq \varepsilon^{r/2}$ and $0 \leq |\chi| < \varepsilon^{r/2}$.

In the former case we have

$$a = \mathcal{O}(\varepsilon^r/|\chi|), \quad b = \sqrt{|\chi|} + \mathcal{O}(\varepsilon^{2r}/|\chi|^{3/2}) = \sqrt{|\chi|} + \mathcal{O}(\varepsilon^r), \quad \sqrt{\chi^2+x^2} = |\chi| + \mathcal{O}(\varepsilon^{3r/2}),$$

whence

$$\begin{aligned}
&\int_{-\varepsilon^{r/2}}^{\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2} \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2+x^2}} d\chi \\
&= \int_{\varepsilon^{r/2}}^{4\pi^2} \frac{\mathcal{O}(\varepsilon^r)(\mathcal{O}(\varepsilon^{2r}/\chi^2) + \mathcal{O}(\chi) + \mathcal{O}(\varepsilon^{3r/2}))}{\chi + \mathcal{O}(\varepsilon^{3r/2})} d\chi = \mathcal{O}(\varepsilon^r) \int_{\varepsilon^{r/2}}^{4\pi^2} [\mathcal{O}(1) + \mathcal{O}(\varepsilon^r/\chi)] d\chi = \mathcal{O}(\varepsilon^r).
\end{aligned}$$

As to the remaining part, using that $a, b \leq (\chi^2+x^2)^{1/4} = \mathcal{O}(\varepsilon^{r/4})$, we have :

$$\int_0^{-\varepsilon^{r/2}} \frac{(b \operatorname{sh} a - a \sin b)(a \operatorname{sh} a + b \sin b - \operatorname{ch} a + \cos b)}{\sqrt{\chi^2+x^2}} d\chi = \mathcal{O}(\varepsilon^r).$$

Therefore we obtain

$$\tilde{\varphi}\left(\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y}x\right) = \frac{5\pi}{4} + \mathcal{O}(\varepsilon^r),$$

whence

$$\varphi\left(\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y}x\right) + \frac{1}{2} \tilde{\varphi}\left(\frac{\varepsilon^2}{\varepsilon-y}-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y}x\right) = -\frac{3\pi}{8} - \frac{1}{2} \operatorname{arctg} x + \mathcal{O}(\varepsilon^{r/3}),$$

whence the claim (of Lemma 8.2.4) follows at once. \diamond

Lemma 8.2.4 allows to handle the part of the integral of Proposition 8.2.3 which is close to the singularity (saddle-point). Indeed, cutting that integral at ε^r we get :

$$\begin{aligned} \int_0^\infty \exp\left[\frac{\varepsilon-y}{\varepsilon^2}\sqrt{-1}x\right] \Phi(4\eta-4\pi^2, x) dx &= \int_{\varepsilon^r}^\infty \exp\left[\frac{\varepsilon-y}{\varepsilon^2}\sqrt{-1}x\right] \Phi(4\eta-4\pi^2, x) dx \\ &+ \frac{\varepsilon^2}{\varepsilon-y} \int_0^{\varepsilon^{r-2}(\varepsilon-y)} \exp[\sqrt{-1}x] \times \Phi\left(4\eta-4\pi^2, \frac{\varepsilon^2 x}{\varepsilon-y}\right) dx. \end{aligned} \quad (46)$$

From Lemma 8.2.4 we deduce the following behavior of the above main contribution to the wanted heat kernel. It will remain to ensure that it is indeed the main one.

Proposition 8.2.5 *For $y \leq 0$, $\varepsilon > 0$ and $r' := \frac{3}{2} - \frac{3}{4} 1_{\{y=0\}} \leq r \leq \frac{9}{10}(2 - 1_{\{y=0\}})$, we have*

$$\begin{aligned} &\Re \left\{ \frac{\varepsilon^2}{\varepsilon-y} \int_0^{\varepsilon^{r-2}(\varepsilon-y)} \exp[\sqrt{-1}x] \times \Phi\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, \frac{\varepsilon^2}{\varepsilon-y} x\right) dx \right\} \\ &= \frac{8\pi^{5/2} \varepsilon}{\sqrt{\varepsilon-y}} \left(1 + \mathcal{O}(\varepsilon^{r/6}) + \mathcal{O}\left(\frac{\varepsilon^{2-r}}{\varepsilon-y}\right)^{3/2}\right) \int_0^\infty \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right)}{(x^2+1)^{1/4}} dx \\ &= \frac{8\pi^{5/2} \varepsilon}{\sqrt{\varepsilon-y}} \left(1 + \mathcal{O}\left(\varepsilon^{\frac{1}{8}(2-1_{\{y=0\}})}\right)\right) \int_0^\infty \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right)}{(x^2+1)^{1/4}} dx \end{aligned}$$

by eventually taking $r = r'$. This holds uniformly with respect to $y \leq 0$.

Proof Replacing $\Phi(4\eta-4\pi^2, \frac{\varepsilon^2}{\varepsilon-y} x)$ according to Lemma 8.2.4, for the above value under consideration we obtain :

$$\begin{aligned} &\frac{\varepsilon}{\sqrt{\varepsilon-y}} \int_0^{\varepsilon^{r-2}(\varepsilon-y)} \frac{8\pi^{5/2} + \mathcal{O}\left(\frac{\varepsilon^2}{\varepsilon-y}\right)}{(x^2+1)^{1/4}} \times \cos\left[x - \frac{3\pi}{8} - \frac{1}{2} \arctg x + \mathcal{O}(\varepsilon^{r/3})\right] dx \\ &= \frac{8\pi^{5/2} \varepsilon + \mathcal{O}\left(\frac{\varepsilon^3}{\varepsilon-y}\right)}{\sqrt{\varepsilon-y}} \int_0^{\varepsilon^{r-2}(\varepsilon-y)} \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x + \mathcal{O}(\varepsilon^{r/3})\right)}{(x^2+1)^{1/4}} dx \\ &= \frac{8\pi^{5/2} \varepsilon + \mathcal{O}\left(\frac{\varepsilon^3}{\varepsilon-y}\right)}{\sqrt{\varepsilon-y}} \int_0^{\varepsilon^r/\eta} \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right) + \mathcal{O}(\varepsilon^{r/3})}{(x^2+1)^{1/4}} dx \quad (\text{recall } \eta = \frac{\varepsilon^2}{4(\varepsilon-y)}) \\ &= \frac{8\pi^{5/2} \varepsilon + \mathcal{O}\left(\frac{\varepsilon^3}{\varepsilon-y}\right)}{\sqrt{\varepsilon-y}} \left[\int_0^{\varepsilon^r/\eta} \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right)}{(x^2+1)^{1/4}} dx + \mathcal{O}(\varepsilon^{r/6}) \right] \quad (\text{as } \eta \geq \varepsilon^{4r/3}) \\ &= \frac{8\pi^{5/2} \varepsilon}{\sqrt{\varepsilon-y}} \left(1 + \mathcal{O}(\varepsilon^{r/6}) + \mathcal{O}(\varepsilon^{-r}\eta)^{3/2}\right) \int_0^\infty \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \arctg x\right)}{(x^2+1)^{1/4}} dx \end{aligned}$$

by Proposition 8.2.6 below. \diamond

Proposition 8.2.6 (i) The constant $\sigma := \int_0^\infty \frac{\sin(\frac{\pi}{8} + x - \frac{1}{2} \arctg x)}{(x^2 + 1)^{1/4}} dx$ is $> \frac{1}{10}$.

(ii) For large R we have $\int_R^\infty \frac{\sin(\frac{\pi}{8} + x - \frac{1}{2} \arctg x)}{(x^2 + 1)^{1/4}} dx = \mathcal{O}(R^{-3/2})$.

Proof (i) Let $\theta(x) := \frac{\pi}{8} + x - \frac{1}{2} \arctg x = x - \frac{\pi}{8} + \frac{1}{2} \arctg \frac{1}{x}$, which increases (from $\frac{\pi}{8}$ to infinity), as well as its derivative θ' and then $\theta' \circ \theta^{-1}$.

Then changing the variable we have:
$$\begin{aligned} \sigma &= \int_{\pi/8}^\infty \frac{\sin t \, dt}{(1 + \theta^{-1}(t)^2)^{1/4} \times \theta' \circ \theta^{-1}(t)} \\ &> \int_{\pi/8}^{\pi/2} \frac{\sin t \, dt}{(1 + \theta^{-1}(\frac{\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{\pi}{2})} + \int_{\pi/2}^\pi \frac{\sin t \, dt}{(1 + \theta^{-1}(\pi)^2)^{1/4} \times \theta' \circ \theta^{-1}(\pi)} \\ &\quad - \int_\pi^{3\pi/2} \frac{|\sin t| \, dt}{(1 + \theta^{-1}(\pi)^2)^{1/4} \times \theta' \circ \theta^{-1}(\pi)} - \int_{3\pi/2}^{2\pi} \frac{|\sin t| \, dt}{(1 + \theta^{-1}(\frac{3\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{3\pi}{2})} \\ &\quad + \sum_{k \geq 1} \left[\int_{2k\pi}^{(2k+1)\pi} \frac{\sin t \, dt}{(1 + \theta^{-1}((2k+1)\pi)^2)^{1/4} \theta' \circ \theta^{-1}((2k+1)\pi)} - \int_{(2k+1)\pi}^{(2k+2)\pi} \frac{|\sin t| \, dt}{(1 + \theta^{-1}((2k+1)\pi)^2)^{1/4} \theta' \circ \theta^{-1}((2k+1)\pi)} \right] \\ &= \int_{\pi/8}^{\pi/2} \frac{\sin t \, dt}{(1 + \theta^{-1}(\frac{\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{\pi}{2})} - \int_{3\pi/2}^{2\pi} \frac{|\sin t| \, dt}{(1 + \theta^{-1}(\frac{3\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{3\pi}{2})} \\ &= \frac{\cos(\pi/8)}{(1 + \theta^{-1}(\frac{\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{\pi}{2})} - \frac{1}{(1 + \theta^{-1}(\frac{3\pi}{2})^2)^{1/4} \times \theta' \circ \theta^{-1}(\frac{3\pi}{2})}. \end{aligned}$$

Now observing the following: $\ast \cos(\pi/8) = \sqrt{\frac{2+\sqrt{2}}{2}} > \frac{92}{100}$;

\ast as $\theta(x) > x - \frac{\pi}{8}$, we have $\theta^{-1}(\frac{\pi}{2}) < \frac{5\pi}{8}$; \ast as $\theta(x) < x - \frac{\pi}{8} + \frac{1}{2x}$, we have $\theta^{-1}(x + \frac{1}{2x + \frac{\pi}{4}}) > x + \frac{\pi}{8}$ and then $\frac{3\pi}{2} > \frac{471}{100} < \frac{9}{2} + \frac{1}{9 + \frac{\pi}{4}} \Rightarrow \theta^{-1}(\frac{3\pi}{2}) > \frac{9}{2} + \frac{\pi}{8} > \frac{3\pi}{2}$;

$\ast \theta'(x) = 1 - \frac{1}{2(1+x^2)}$, and then $(1 + \theta^{-1}(t)^2)^{1/4} \times \theta' \circ \theta^{-1}(t) = \frac{1+2\theta^{-1}(t)^2}{2(1+\theta^{-1}(t)^2)^{3/4}}$,

this yields:

$$\sigma > \frac{92 \times 2 \times (1 + \frac{25\pi^2}{64})^{3/4}}{100 \times (1 + \frac{25\pi^2}{32})} - \frac{2 \times (1 + \frac{9\pi^2}{4})^{3/4}}{1 + \frac{9\pi^2}{2}}.$$

Using $\frac{25\pi^2}{64} < \frac{386}{100}$ and $\frac{9\pi^2}{4} > \frac{222}{10}$, this gives

$$\sigma > \frac{184 \times (4.86)^{3/4}}{872} - \frac{2 \times (23.2)^{3/4}}{45.4} > \frac{1.8 \times 27}{88} - \frac{22}{45.4} > \frac{1}{10}.$$

(ii) Performing the change $x = \cotg \theta = \tg(\frac{\pi}{2} - \theta)$ we get:

$$\int_R^\infty \frac{\sin(\frac{\pi}{8} + x - \frac{1}{2} \arctg x)}{(x^2 + 1)^{1/4}} dx = \int_0^{\arctg(1/R)} \sin[\frac{1}{2} \theta - \frac{\pi}{8} + \cotg \theta] (\sin \theta)^{-\frac{3}{2}} d\theta$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{\arctg \frac{1}{R}} \sin\left[\frac{\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \sqrt{\sin \theta} d\theta + \int_0^{\arctg \frac{1}{R}} \frac{d}{d\theta} \left\{ \cos\left[\frac{\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \right\} \sqrt{\sin \theta} d\theta \\
&= \mathcal{O}(R^{-3/2}) + \frac{1}{2} \int_0^{\arctg \frac{1}{R}} \left(\sin\left[\frac{\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \sin \theta - \cos\left[\frac{\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \cos \theta \right) \frac{d\theta}{\sqrt{\sin \theta}} \\
&= \mathcal{O}(R^{-3/2}) - \frac{1}{2} \int_0^{\arctg \frac{1}{R}} \cos\left[\frac{3\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \frac{d\theta}{\sqrt{\sin \theta}} \\
&= \mathcal{O}(R^{-3/2}) - \frac{1}{2} \int_0^{\arctg \frac{1}{R}} \left(\frac{3}{2} \cos\left[\frac{3\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] - \frac{d}{d\theta} \left\{ \sin\left[\frac{3\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \right\} \right) (\sin \theta)^{3/2} d\theta \\
&= \mathcal{O}(R^{-3/2}) - \frac{3}{4} \int_0^{\arctg \frac{1}{R}} \sin\left[\frac{3\theta}{2} - \frac{\pi}{8} + \cotg \theta\right] \cos \theta \sqrt{\sin \theta} d\theta = \mathcal{O}(R^{-3/2}). \quad \diamond
\end{aligned}$$

To conclude the first sub-case $[y \leq 0, z = 0]$ we are dealing with, we have now to handle the remaining integral in (46), namely $\int_{\varepsilon^{r'}}^{\infty} e^{\sqrt{-1} \frac{\varepsilon-y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx$ (recall that r' was defined in Proposition 8.2.5).

Lemma 8.2.7 *We uniformly have*

$$\begin{aligned}
&\int_{\varepsilon^{r'}}^{\infty} \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(4\eta - 4\pi^2, x) dx \\
&= \frac{\varepsilon}{\sqrt{\varepsilon-y}} \times o(\varepsilon^{\frac{1}{20}}) + \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon-y} \int_{\varepsilon^{r'}}^{9 \log^2 \varepsilon} \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right] \frac{d\Phi}{dx}\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, x\right) dx.
\end{aligned}$$

Proof Lemma 8.2.2 provides the uniform estimate: $\Phi\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, x\right) = \frac{\mathcal{O}(x)}{\sqrt{\text{ch} \sqrt{x/2}}}$ for large x , so that we have

$$\begin{aligned}
\int_R^{\infty} e^{\sqrt{-1} \frac{\varepsilon-y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx &= \mathcal{O}\left[\int_{\sqrt{R/2}}^{\infty} \frac{x^3}{\sqrt{\text{ch} x}} dx\right] = \mathcal{O}\left[\int_{\sqrt{R/8}}^{\infty} x^3 e^{-x} dx\right] \\
&= \mathcal{O}\left[R^{3/2} e^{-\sqrt{R/8}}\right],
\end{aligned}$$

and then

$$\int_{9 \log^2 \varepsilon}^{\infty} e^{\sqrt{-1} \frac{\varepsilon-y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx = \mathcal{O}\left[\varepsilon^{3/2\sqrt{2}} \log^3\left(\frac{1}{\varepsilon}\right)\right] = o\left(\varepsilon^{1+\frac{1}{20}}\right) = \frac{\varepsilon}{\sqrt{\varepsilon-y}} \times o(\varepsilon^{\frac{1}{20}}).$$

It remains to control $\int_{\varepsilon^{r'}}^{9 \log^2 \varepsilon} e^{\sqrt{-1} \frac{\varepsilon-y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx$. Integrating by parts we have:

$$\begin{aligned}
\int_{\varepsilon^{r'}}^{9 \log^2 \varepsilon} \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right] \Phi\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, x\right) dx &= \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon-y} \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right] \Phi\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, \varepsilon^{r'}\right) \\
&\quad - \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon-y} \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} 9 \log^2 \varepsilon\right] \Phi\left(\frac{\varepsilon^2}{\varepsilon-y} - 4\pi^2, 9 \log^2 \varepsilon\right)
\end{aligned}$$

$$+ \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon - y} \int_{\varepsilon^{r'}}^{\varepsilon^{9 \log^2 \varepsilon}} \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \frac{d\Phi}{dx} \left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x \right) dx.$$

Now, according to Lemma 8.2.4 we uniformly have $\Phi(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, \varepsilon^{r'}) = \mathcal{O}(\varepsilon^{-r'/2})$, so that (noticing that $\eta \asymp \varepsilon^{4r'/3}$) the first term of the right hand side is $\frac{\varepsilon \times \mathcal{O}(\varepsilon^{r'/6})}{\sqrt{\varepsilon - y}}$. Moreover by Lemma 8.2.2 we have $\Phi(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, 9 \log^2 \varepsilon) = \mathcal{O}[\varepsilon^{3/\sqrt{2}} \log^2(\frac{1}{\varepsilon})] = o(\varepsilon^2)$, so that the second term of the right hand side is $\frac{\varepsilon \times o(\varepsilon^2)}{\sqrt{\varepsilon - y}}$. Finally we have $\mathcal{O}(\varepsilon^{r'/6}) = o(\varepsilon^{\frac{1}{20}})$. \diamond

Then Lemma 8.2.2 allows to estimate $\frac{d\Phi}{dx}(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x)$. The following will be sufficient.

Lemma 8.2.8 *For any $y \leq 0$ and $x \geq \varepsilon^{r'}$, uniformly as $\varepsilon \searrow 0$ we have*

$$\frac{d\Phi}{dx}(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x) = 1_{\{\varepsilon^{r'} \leq x < \varepsilon^{2r'/3}\}} \frac{\mathcal{O}(\varepsilon^{4r'/3})}{x^{5/2}} + 1_{\{\varepsilon^{2r'/3} \leq x < 1\}} \frac{\mathcal{O}(1)}{\sqrt{x}} + 1_{\{x \geq 1\}} \mathcal{O}(x e^{-\sqrt{x/8}}).$$

Proof As already used above, Lemma 8.2.4 ensures that $\Phi(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x) = \mathcal{O}(x^{-1/2})$ for small x and Lemma 8.2.2 ensures that $\Phi(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x) = \mathcal{O}[x e^{-\sqrt{x/8}}]$ for large x , uniformly with respect to $y \leq 0$. Then using Lemma 8.2.2 again and (37) we obtain

$$\begin{aligned} \frac{d \log \Phi}{dx}(\chi, x) &= \sqrt{-1} \frac{d(\varphi + \frac{1}{2} \tilde{\varphi})}{dx}(\chi, x) + \frac{5x}{4\sqrt{\chi^2 + x^2}} + \frac{a \sin b - b \operatorname{sh} a}{8(a^2 + b^2)(\operatorname{ch} a - \cos b)} \\ &\quad + \frac{a \sin b - b \operatorname{sh} a}{8((a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{ch} a + b \sin b) + 4(\operatorname{ch} a - \cos b))} \\ &= \sqrt{-1} \frac{d(\varphi + \frac{1}{2} \tilde{\varphi})}{dx}(\chi, x) + \mathcal{O}(1). \end{aligned}$$

Then by (37) we have $a \frac{\partial a}{\partial x} = b \frac{\partial b}{\partial x} = \frac{x}{4\sqrt{\chi^2 + x^2}} = \mathcal{O}(1)$, $\frac{\partial a}{\partial x} = \frac{b/2}{a^2 + b^2}$, $\frac{\partial b}{\partial x} = \frac{a/2}{a^2 + b^2}$,

and by Lemma 7.4 we have

$$\frac{d\varphi}{dx}(\chi, x) = \frac{1}{2} \frac{d}{dx} [\operatorname{arctg}(b/a) - \varphi_a(b)] = \frac{\chi}{4(\chi^2 + x^2)} - \frac{a \operatorname{sh}(2a) - b \sin(2b)}{4(a^2 + b^2)(\operatorname{ch}(2a) - \cos(2b))}$$

which is bounded for $x \geq 1$; while for small x , proceeding as in the proof of Lemma 8.2.4 we successively have $\operatorname{ch}(2a) - \cos(2b) \sim \frac{x^2}{8\pi^2}$, $a \operatorname{sh}(2a) - b \sin(2b) \sim 2\eta + \frac{3x^2}{32\pi^2}$, whence

$$\frac{d\varphi}{dx}(\chi, x) = \mathcal{O}(1 + \eta x^{-2}) = 1_{\{\varepsilon^{r'} \leq x < \varepsilon^{2r'/3}\}} \mathcal{O}(\varepsilon^{4r'/3}/x^2) + 1_{\{x \geq \varepsilon^{2r'/3}\}} \mathcal{O}(1).$$

Then by Lemma 8.2.1 and (38) we have

$$\frac{d\tilde{\varphi}}{dx}(\chi, x) = \frac{d}{dx} \arg \left[\frac{\chi + \sqrt{-1} x}{f(\chi, x)} \right] = \frac{d}{dx} \operatorname{arctg} \frac{x}{\chi} - \frac{d}{dx} \arg \left[1 - \frac{a \operatorname{sh} a + b \sin b + \sqrt{-1} (a \sin b - b \operatorname{sh} a)}{(a^2 + b^2)(\operatorname{ch} a + \cos b)/2} \right]$$

$$\begin{aligned}
&= \frac{\chi}{\chi^2 + x^2} - \frac{d}{dx} \operatorname{arctg} \left[\frac{2(a \sin b - b \operatorname{sh} a)}{2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)} \right] \\
&= \frac{\chi}{\chi^2 + x^2} - \frac{(b \partial_a(a \sin b - b \operatorname{sh} a) + a \partial_b(a \sin b - b \operatorname{sh} a)) [2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)]}{(a^2 + b^2) \left([2(a \sin b - b \operatorname{sh} a)]^2 + [2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)]^2 \right)} \\
&\quad + \frac{(b \partial_a[2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)] + a \partial_b[2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)]) (a \sin b - b \operatorname{sh} a)}{(a^2 + b^2) \left([2(a \sin b - b \operatorname{sh} a)]^2 + [2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)]^2 \right)} \\
&= \frac{\chi}{\chi^2 + x^2} + \frac{\mathcal{O}((a^2 + b^2)^2 (\operatorname{ch} a + \cos b)^2)}{(a^2 + b^2)^2 (\operatorname{ch} a + \cos b) [(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)]}
\end{aligned}$$

which is bounded. The claim follows. \diamond

The following actually states that the main contribution to the wanted heat kernel was indeed given by the part considered in Proposition 8.2.5 ; in other words, that the dominant saddle point of the oscillatory integral $\mathcal{I}_\varepsilon(y, 0)$ was indeed located near $4\sqrt{-1} \left(\pi^2 - \frac{\varepsilon^2}{4(\varepsilon - y)} \right)$.

Lemma 8.2.9 *We uniformly have $\int_{\varepsilon^{r'}}^{\infty} e^{\sqrt{-1} \frac{\varepsilon - y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx = \frac{\varepsilon}{\sqrt{\varepsilon - y}} \times o(\varepsilon^{\frac{1}{20}})$.*

Proof By Lemmas 8.2.7 and 8.2.8 (recalling $\eta \asymp \varepsilon^{4r'/3}$) we have :

$$\begin{aligned}
&\int_{\varepsilon^{r'}}^{\infty} e^{\sqrt{-1} \frac{\varepsilon - y}{\varepsilon^2} x} \Phi(4\eta - 4\pi^2, x) dx - \frac{\varepsilon}{\sqrt{\varepsilon - y}} \times o(\varepsilon^{\frac{1}{20}}) \\
&= \frac{\varepsilon \mathcal{O}(\varepsilon^{2r'/3})}{\sqrt{\varepsilon - y}} \left[\int_{\varepsilon^{r'}}^{\varepsilon^{2r'}} \frac{\mathcal{O}(\varepsilon^{4r'/3})}{x^{5/2}} dx + \int_{\varepsilon^{2r'}}^1 \frac{\mathcal{O}(1)}{\sqrt{x}} dx + \int_1^{9 \log^2 \varepsilon} \mathcal{O}(x e^{-\sqrt{x/8}}) dx \right] \\
&= \frac{\varepsilon \mathcal{O}(\varepsilon^{2r'/3})}{\sqrt{\varepsilon - y}} \left[\varepsilon^{4r'/3 - 3r'/2} + 1 \right] = \frac{\varepsilon}{\sqrt{\varepsilon - y}} \times \mathcal{O}(\varepsilon^{r'/2}) = \frac{\varepsilon}{\sqrt{\varepsilon - y}} \times o(\varepsilon^{\frac{1}{20}}). \quad \diamond
\end{aligned}$$

We can now conclude this section, by the following exact equivalent of the oscillatory integral $\mathcal{I}_\varepsilon(y, 0)$ arising in the case $y \leq 0 = z$.

Proposition 8.2.10 *For $y \leq 0$, uniformly as $\varepsilon \searrow 0$ we have*

$$\Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \times \Phi \left(\frac{\varepsilon^2}{\varepsilon - y} - 4\pi^2, x \right) dx \right\} = \frac{8\pi^{5/2} \sigma \varepsilon}{\sqrt{\varepsilon - y}} \left(1 + o(\varepsilon^{\frac{1}{20}}) \right)$$

(recall σ is that of Proposition 8.2.6) and

$$\mathcal{I}_\varepsilon(y, 0) = \exp \left[4\pi^2 \frac{y - \varepsilon}{\varepsilon^2} \right] \times \frac{\varepsilon}{\sqrt{\varepsilon - y}} \times 16 \pi^{5/2} e \sigma \times \left(1 + o(\varepsilon^{\frac{1}{20}}) \right).$$

Proof The first claim follows directly from (46) with $\eta = \frac{\varepsilon^2}{4(\varepsilon-y)}$ and $r = r' = \frac{3}{2} - \frac{3}{4} 1_{\{y=0\}}$, Lemma 8.2.9 and Proposition 8.2.5. With Proposition 8.2.3, it entails the second one. \diamond

Propositions 6.3.1 and 8.2.10 together give the following wanted small time equivalent, which is the content of Theorem 2.1(ii).

Corollary 8.2.11 *For $y \leq 0$, uniformly as $\varepsilon \searrow 0$ we have*

$$p_\varepsilon(0; (0, y, 0)) \sim \exp\left[4\pi^2 \frac{y - \varepsilon}{\varepsilon^2}\right] \times \frac{2\sqrt{2} e}{\varepsilon^3 \sqrt{\varepsilon - y}} \times \int_0^\infty \frac{\sin\left(\frac{\pi}{8} + x - \frac{1}{2} \operatorname{arctg} x\right)}{(x^2 + 1)^{1/4}} dx.$$

8.3 Second sub-case: $y > 0$, $z = 0$

Having dealed in Section 8.2 with the sub-case $y \leq 0$, we deal here with the sub-case $y > 0$, in order (recall Proposition 8.1.2) to handel

$$\mathcal{I}_\varepsilon(y, 0) = 2 \Re \left\{ \int_0^\infty \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(x) dx \right\}.$$

The dominant saddle point of the above oscillatory integral, not easy to compute, happens to be now close to $-\sqrt{-1} \pi^2$ (very roughly: because π^2 is the first positive zero of $(\operatorname{ch} \sqrt{-2\sqrt{-1} x} - \cos \sqrt{-2\sqrt{-1} x})$), as will be confirmed later, by Proposition 8.3.7. Since it is located under the real axis, the situation will be partly simpler as in Section 8.2 (Corollary 7.5 lets guess that the case of a positive χ should be a priori easier). We however follow the same route, using part of the work already made in this former case.

To use that $-\sqrt{-1} \pi^2$ is indeed close to a saddle point, we need to expand Φ partially. Here is the analogue of Lemma 8.2.4.

Lemma 8.3.1 *For small h , we have $\Phi(\pi^2, h) \equiv \Phi(-\sqrt{-1} \pi^2 + h) =$*

$$\frac{2^{3/4} \pi^{11/4} e^{5\sqrt{-1} \pi/16}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times h^{-1/4} \times \exp \left[\sqrt{-1} \left[\frac{5/4 - 2\pi \coth \pi - \pi \operatorname{th} \pi}{8\pi^2} + \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{4\pi^2 (\pi - 2 \operatorname{th} \frac{\pi}{2})} \right] h + \mathcal{O}(h^2) \right].$$

Proof To begin, let us directly use the expression for Φ given in Proposition 8.1.2, applied first with $\sqrt{\frac{x}{2}} = (1 - \sqrt{-1}) \frac{\pi}{2} + u$, i.e., with $x = -\sqrt{-1} \pi^2 + 2(1 - \sqrt{-1}) \pi u + u^2$ (by parity, considering $-\sqrt{\frac{x}{2}}$ instead of $\sqrt{\frac{x}{2}}$ would yield the same).

For that we need the following approximations.

$$\begin{aligned} \operatorname{ch}[(1 - \sqrt{-1}) \frac{\pi}{2}] &= -\sqrt{-1} \operatorname{sh} \frac{\pi}{2} ; \quad \cos[(1 - \sqrt{-1}) \frac{\pi}{2}] = \sqrt{-1} \operatorname{sh} \frac{\pi}{2} ; \quad \operatorname{sh}[(1 - \sqrt{-1}) \frac{\pi}{2}] = -\sqrt{-1} \operatorname{ch} \frac{\pi}{2} ; \\ \sin[(1 - \sqrt{-1}) \frac{\pi}{2}] &= \operatorname{ch} \frac{\pi}{2} ; \quad \operatorname{ch} \sqrt{\frac{x}{2}} = -\sqrt{-1} \operatorname{sh} \frac{\pi}{2} (1 + u \coth \frac{\pi}{2} + \frac{1}{2} u^2 + \mathcal{O}(u^3)) ; \\ \operatorname{sh} \sqrt{\frac{x}{2}} &= -\sqrt{-1} \operatorname{ch} \frac{\pi}{2} (1 + u \operatorname{th} \frac{\pi}{2} + \frac{1}{2} u^2 + \mathcal{O}(u^3)) ; \quad \sin \sqrt{\frac{x}{2}} = \operatorname{ch} \frac{\pi}{2} (1 + u \sqrt{-1} \operatorname{th} \frac{\pi}{2} - \frac{1}{2} u^2 + \mathcal{O}(u^3)) ; \end{aligned}$$

$$\begin{aligned}
\cos\sqrt{\frac{x}{2}} &= \sqrt{-1} \operatorname{sh} \frac{\pi}{2} \left(1 + u\sqrt{-1} \coth \frac{\pi}{2} - \frac{1}{2}u^2 + \mathcal{O}(u^3)\right); \\
\operatorname{ch}\sqrt{2x} - \cos\sqrt{2x} &= 2(\operatorname{ch}^2\sqrt{\frac{x}{2}} - \cos^2\sqrt{\frac{x}{2}}) = 2(\sqrt{-1} - 1)u \operatorname{sh} \pi - 4u^2 \operatorname{ch} \pi + \mathcal{O}(u^3); \\
\operatorname{ch}\sqrt{\frac{x}{2}} + \cos\sqrt{\frac{x}{2}} &= -(1 + \sqrt{-1})u \operatorname{ch} \frac{\pi}{2} - u^2\sqrt{-1} \operatorname{sh} \frac{\pi}{2} + \mathcal{O}(u^3); \\
\operatorname{sh}\sqrt{\frac{x}{2}} - \sin\sqrt{\frac{x}{2}} &= -\sqrt{-1} \operatorname{ch} \frac{\pi}{2} \left((1 - \sqrt{-1}) + 2u \operatorname{th} \frac{\pi}{2} + \frac{1}{2}(1 + \sqrt{-1})u^2 + \mathcal{O}(u^3)\right); \\
\operatorname{sh}\sqrt{\frac{x}{2}} + \sin\sqrt{\frac{x}{2}} &= -\sqrt{-1} \operatorname{ch} \frac{\pi}{2} \left((1 + \sqrt{-1}) + \frac{1}{2}(1 - \sqrt{-1})u^2 + \mathcal{O}(u^3)\right).
\end{aligned}$$

Therefore on the one hand we have

$$\begin{aligned}
\left[\frac{2x}{\operatorname{ch}\sqrt{2x} - \cos\sqrt{2x}}\right]^{1/4} &= \left[\frac{(\sqrt{-1} - 1) \operatorname{sh} \pi u - 2 \operatorname{ch} \pi u^2 + \mathcal{O}(u^3)}{-\sqrt{-1} \pi^2 + 2(1 - \sqrt{-1})\pi u + u^2}\right]^{-1/4} \\
&= \left[-\frac{1 + \sqrt{-1}}{\pi^2} \times \frac{\operatorname{sh} \pi u + (1 + \sqrt{-1}) \operatorname{ch} \pi u^2 + \mathcal{O}(u^3)}{1 + 2(1 + \sqrt{-1})u/\pi + \sqrt{-1} u^2/\pi^2}\right]^{-1/4} \\
&= \sqrt{\pi} 2^{-1/8} e^{3\sqrt{-1}\pi/8} (u \operatorname{sh} \pi)^{-1/4} \left[1 - \frac{1+\sqrt{-1}}{4} (\operatorname{th} \pi - \frac{2}{\pi}) u + \mathcal{O}(u^2)\right] \\
&= \sqrt{\pi} 2^{-1/8} e^{3\sqrt{-1}\pi/8} (u \operatorname{sh} \pi)^{-1/4} \exp\left[-\frac{1+\sqrt{-1}}{4} (\operatorname{th} \pi - \frac{2}{\pi}) u + \mathcal{O}(u^2)\right].
\end{aligned}$$

Then on the other hand we get :

$$\begin{aligned}
\frac{\sqrt{-1}x}{f(0, x)} &= \frac{\left[-\sqrt{-1}\pi^2 + 2(1 - \sqrt{-1})\pi u + u^2\right] \left[(1 - \sqrt{-1})\frac{\pi}{2} + u\right] \left[-(1 + \sqrt{-1})u \operatorname{ch} \frac{\pi}{2} - u^2\sqrt{-1} \operatorname{sh} \frac{\pi}{2} + \mathcal{O}(u^3)\right]}{-2\sqrt{-1} \operatorname{sh} \frac{\pi}{2} u + (1 - \sqrt{-1}) \operatorname{ch} \frac{\pi}{2} u^2 + \mathcal{O}(u^3) - \sqrt{-1} \left[(1 - \sqrt{-1})\frac{\pi}{2} + u\right] \left[-(1 + \sqrt{-1})u \operatorname{ch} \frac{\pi}{2} - u^2\sqrt{-1} \operatorname{sh} \frac{\pi}{2}\right]} \\
&= \frac{\left[\pi^2 + 2(1 + \sqrt{-1})\pi u + \sqrt{-1}u^2\right] \left[(1 + \sqrt{-1})\frac{\pi}{2} + \sqrt{-1}u\right] \left[(1 + \sqrt{-1}) \operatorname{ch} \frac{\pi}{2} + u\sqrt{-1} \operatorname{sh} \frac{\pi}{2} + \mathcal{O}(u^2)\right]}{-2\sqrt{-1} \operatorname{sh} \frac{\pi}{2} + (1 - \sqrt{-1}) \operatorname{ch} \frac{\pi}{2} u + \mathcal{O}(u^2) + \left[(1 + \sqrt{-1})\frac{\pi}{2} + \sqrt{-1}u\right] \left[(1 + \sqrt{-1}) \operatorname{ch} \frac{\pi}{2} + u\sqrt{-1} \operatorname{sh} \frac{\pi}{2}\right]} \\
&= \frac{\pi^2 \left[\pi \operatorname{ch} \frac{\pi}{2} + (1 + \sqrt{-1})(3 \operatorname{ch} \frac{\pi}{2} + \frac{\pi}{2} \operatorname{sh} \frac{\pi}{2})u + \mathcal{O}(u^2)\right]}{(\pi \operatorname{ch} \frac{\pi}{2} - 2 \operatorname{sh} \frac{\pi}{2}) + (1 + \sqrt{-1})\frac{\pi}{2} \operatorname{sh} \frac{\pi}{2} u + \mathcal{O}(u^2)} \\
&= \frac{\pi^3}{\pi - 2 \operatorname{th} \frac{\pi}{2}} \times \left[1 + (1 + \sqrt{-1}) \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{\pi(\pi - 2 \operatorname{th} \frac{\pi}{2})} u + \mathcal{O}(u^2)\right],
\end{aligned}$$

so that

$$\sqrt{\frac{2\pi\sqrt{-1}x}{f(0, x)}} = \frac{\sqrt{2}\pi^2}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}}} \times \exp\left[(1 + \sqrt{-1}) \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{2\pi(\pi - 2 \operatorname{th} \frac{\pi}{2})} u + \mathcal{O}(u^2)\right].$$

Moreover, according to Lemma 7.4 and setting $v := 2\pi(1 - \sqrt{-1})u + u^2$ we have

$$\varphi(x) = \frac{1}{2} \operatorname{arctg} \sqrt{\frac{\sqrt{\pi^4 + v^2} - \pi^2}{\sqrt{\pi^4 + v^2} + \pi^2}} - \frac{1}{2} \operatorname{arctg} \left(\operatorname{tg} \sqrt{\frac{\sqrt{\pi^4 + v^2} - \pi^2}{2}} \times \coth \sqrt{\frac{\sqrt{\pi^4 + v^2} + \pi^2}{2}} \right)$$

$$\begin{aligned}
&= \frac{1}{2} \operatorname{arctg} \left[\frac{v}{2\pi^2} - \frac{v^3}{8\pi^6} + \mathcal{O}(v^5) \right] - \frac{1}{2} \operatorname{arctg} \left[\operatorname{tg} \left[\frac{v}{2\pi} - \frac{v^3}{16\pi^5} + \mathcal{O}(v^5) \right] \coth \left[\pi + \frac{v^2}{8\pi^3} + \mathcal{O}(v^4) \right] \right] \\
&= \frac{1}{2} \operatorname{arctg} \left[\frac{v}{2\pi^2} - \frac{v^3}{8\pi^6} + \mathcal{O}(v^5) \right] - \frac{1}{2} \operatorname{arctg} \left[\frac{v}{2\pi} \coth \pi + \frac{(2\pi^2 - 3) \operatorname{ch} \pi \operatorname{sh} \pi - 3\pi}{48 \pi^5 \operatorname{sh}^2 \pi} v^3 + \mathcal{O}(v^5) \right] \\
&= \frac{1 - \pi \coth \pi}{4\pi^2} v - \frac{(2\pi^2 - 3) \pi \operatorname{ch} \pi \operatorname{sh} \pi - 3\pi^2 + (8 - 2\pi^3) \operatorname{sh}^2 \pi}{96 \pi^6 \operatorname{sh}^2 \pi} v^3 + \mathcal{O}(v^5) \\
&= \frac{1 - \pi \coth \pi}{2\pi} (1 - \sqrt{-1}) u + \frac{1 - \pi \coth \pi}{4\pi^2} u^2 \\
&\quad + \frac{(2\pi^2 - 3) \pi \operatorname{ch} \pi \operatorname{sh} \pi - 3\pi^2 + (8 - 2\pi^3) \operatorname{sh}^2 \pi}{6 \pi^3 \operatorname{sh}^2 \pi} (1 + \sqrt{-1}) u^3 + \mathcal{O}(v^4).
\end{aligned}$$

Hence, according to Proposition 8.1.2, for $x = -\sqrt{-1} \pi^2 + 2(1 - \sqrt{-1}) \pi u + u^2$ we have :

$$\begin{aligned}
\Phi(x) &= e^{\sqrt{-1} \varphi(x)} \times \left[\frac{2x}{\operatorname{ch} \sqrt{2x} - \cos \sqrt{2x}} \right]^{1/4} \times \sqrt{\frac{2\pi \sqrt{-1} x}{f(0, x)}} \\
&= \frac{2^{3/8} \pi^{5/2} e^{3\sqrt{-1} \pi/8}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (u \operatorname{sh} \pi)^{1/4}} \times \exp \left[(1 + \sqrt{-1}) \left(\frac{1 - \pi \coth \pi}{2\pi} + \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{2\pi(\pi - 2 \operatorname{th} \frac{\pi}{2})} - \frac{\pi \operatorname{th} \pi - 2}{4\pi} \right) u + \mathcal{O}(u^2) \right] \\
&= \frac{2^{3/8} \pi^{5/2} e^{3\sqrt{-1} \pi/8}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (u \operatorname{sh} \pi)^{1/4}} \exp \left[(1 + \sqrt{-1}) \left(\frac{4 - 2\pi \coth \pi - \pi \operatorname{th} \pi}{4\pi} + \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{2\pi(\pi - 2 \operatorname{th} \frac{\pi}{2})} \right) u + \mathcal{O}(u^2) \right].
\end{aligned}$$

Equivalently, taking $h = 2(1 - \sqrt{-1}) \pi u + u^2 \Leftrightarrow u = \frac{1 + \sqrt{-1}}{4\pi} (1 - \frac{\sqrt{-1} h}{8\pi^2} + \mathcal{O}(h^2)) h$, this yields

$$\begin{aligned}
\Phi(\pi^2, h) &= \frac{2^{3/4} \pi^{11/4} e^{5\sqrt{-1} \pi/16}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \left(\frac{1 + \frac{\sqrt{-1} h}{32\pi^2} + \mathcal{O}(h^2)}{h^{1/4}} \right) \times \\
&\quad \times \exp \left[\sqrt{-1} \left(\frac{4 - 2\pi \coth \pi - \pi \operatorname{th} \pi}{8\pi^2} + \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{4\pi^2(\pi - 2 \operatorname{th} \frac{\pi}{2})} \right) h + \mathcal{O}(h^2) \right] = \\
&= \frac{2^{3/4} \pi^{11/4} e^{5\sqrt{-1} \pi/16}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} h^{-1/4} \times \exp \left[\sqrt{-1} \left[\frac{5/4 - 2\pi \coth \pi - \pi \operatorname{th} \pi}{8\pi^2} + \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{4\pi^2(\pi - 2 \operatorname{th} \frac{\pi}{2})} \right] h + \mathcal{O}(h^2) \right]. \diamond
\end{aligned}$$

For the analogue of Proposition 8.2.3, let us consider the new contour

$$\Gamma := [0, -\sqrt{-1} \pi^2] \cup (-\sqrt{-1} \pi^2 + \mathbb{R}_+). \quad (47)$$

Proposition 8.3.2 *For any $y > 0$ and small $\varepsilon > 0$ we have*

$$\mathcal{I}_\varepsilon(y, 0) = \exp \left[-\frac{y - \varepsilon}{\varepsilon^2} \pi^2 \right] \times 2 \Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x \right] \Phi(\pi^2, x) dx \right\},$$

where $\Phi(\chi, x) \equiv \Phi(x - \sqrt{-1} \chi)$ was defined in (41).

Proof The change of contour (47) is justified as for Proposition 8.2.3, using $|\Phi(\chi, R)| = \mathcal{O}(R e^{-\sqrt{R/8}})$ as well. Performing this change yields :

$$\begin{aligned} \int_0^\infty \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(x) dx &= \int_\Gamma \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(x) dx = \\ &\exp\left[\frac{\varepsilon - y}{\varepsilon^2} \pi^2\right] \times \int_0^\infty \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, x) dx - \sqrt{-1} \int_0^{\pi^2} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} x\right] \Phi(x, 0) dx. \end{aligned}$$

Then for any $\chi > 0$, we have $\varphi(\chi, 0) = 0$ by Lemma 7.4, and $f(\chi, 0) \in \mathbb{R}$ by (38), and then $\Phi(\chi, 0) \in \mathbb{R}$ by (41). Thus the second term of the right hand side disappears when taking the real part. \diamond

The analogue of (46) is the following.

$$\begin{aligned} \int_0^\infty \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, x) dx &= \int_{\varepsilon^r}^\infty \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, x) dx \\ &+ \frac{\varepsilon^2}{y - \varepsilon} \int_0^{\varepsilon^{r-2}(y-\varepsilon)} \exp[-\sqrt{-1} x] \times \Phi\left(\pi^2, \frac{\varepsilon^2 x}{y - \varepsilon}\right) dx. \end{aligned} \quad (48)$$

We now apply Lemma 8.3.1 to the last integral in (48), to obtain the following analogue of Proposition 8.2.5.

Proposition 8.3.3 *For $y > 0$, $\varepsilon > 0$ and $0 < r < 2$, we have*

$$\begin{aligned} &\Re\left\{ \int_0^{\varepsilon^{r-2}(y-\varepsilon)} \exp[-\sqrt{-1} x] \times \Phi\left(\pi^2, \frac{\varepsilon^2 x}{y - \varepsilon}\right) dx \right\} \\ &= \frac{2^{3/4} \pi^{11/4} y^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{1 + \mathcal{O}(\varepsilon^r + \varepsilon^{(2-r)/4})}{\sqrt{\varepsilon}} \int_0^\infty \frac{\sin(\frac{3\pi}{16} + x)}{x^{1/4}} dx \\ &= \frac{2^{3/4} \pi^{11/4} y^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{1 + \mathcal{O}(\varepsilon^{2/5})}{\sqrt{\varepsilon}} \int_0^\infty \frac{\sin(\frac{3\pi}{16} + x)}{x^{1/4}} dx \quad \text{by eventually taking } r = 2/5. \end{aligned}$$

Proof Applying Lemma 8.3.1 we indeed have :

$$\begin{aligned} &\Re\left\{ \int_0^{\varepsilon^{r-2}(y-\varepsilon)} \exp[-\sqrt{-1} x] \times \Phi\left(\pi^2, \frac{\varepsilon^2 x}{y - \varepsilon}\right) dx \right\} = \frac{2^{3/4} \pi^{11/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \\ &\quad \times \int_0^{\varepsilon^{r-2}(y-\varepsilon)} \left(\frac{\varepsilon^2 x}{y - \varepsilon}\right)^{-1/4} \Re\left\{ \exp\left[-\sqrt{-1} x + \sqrt{-1} \frac{5\pi}{16} + \mathcal{O}(\varepsilon^r)\right] \right\} dx \\ &= \frac{2^{3/4} \pi^{11/4} (y - \varepsilon)^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{1 + \mathcal{O}(\varepsilon^r)}{\sqrt{\varepsilon}} \int_0^{\varepsilon^{r-2}(y-\varepsilon)} \cos\left[x - \frac{5\pi}{16}\right] x^{-1/4} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2^{3/4} \pi^{11/4} y^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{1 + \mathcal{O}(\varepsilon^r + \varepsilon)}{\sqrt{\varepsilon}} \left[\int_0^\infty \sin \left[x + \frac{3\pi}{16} \right] x^{-1/4} dx + \mathcal{O}(\varepsilon^{(2-r)/4}) \right] \\
&= \frac{2^{3/4} \pi^{11/4} y^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{1 + \mathcal{O}(\varepsilon^r + \varepsilon^{(2-r)/4})}{\sqrt{\varepsilon}} \int_0^\infty \sin \left[x + \frac{3\pi}{16} \right] x^{-1/4} dx
\end{aligned}$$

by the following (easy analogue of Proposition 8.2.6). \diamond

Lemma 8.3.4 *The constant $\sigma' := \int_0^\infty \frac{\sin(\frac{3\pi}{16} + x)}{x^{1/4}} dx$ is positive ($> \frac{1}{10}$).*

Proof We indeed have

$$\begin{aligned}
\sigma' &= \int_{\frac{3\pi}{16}}^\infty \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} = \int_{\frac{3\pi}{16}}^{\frac{\pi}{2}} \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} + \int_{\frac{\pi}{2}}^\pi \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} + \int_\pi^{\frac{3\pi}{2}} \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} \\
&\quad + \int_{\frac{3\pi}{2}}^{2\pi} \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} + \sum_{n \geq 1} \left[\int_{2n\pi}^{(2n+1)\pi} \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} + \int_{(2n+1)\pi}^{2(n+1)\pi} \frac{\sin x dx}{(x - \frac{3\pi}{16})^{1/4}} \right] \\
&> \left(\frac{5\pi}{16}\right)^{-1/4} \cos\left(\frac{3\pi}{16}\right) + \left(\frac{13\pi}{16}\right)^{-1/4} - \left(\frac{13\pi}{16}\right)^{-1/4} - \left(\frac{21\pi}{16}\right)^{-1/4} = \left(\frac{16}{5\pi}\right)^{1/4} \left[\cos\left(\frac{3\pi}{16}\right) - \left(4 + \frac{1}{5}\right)^{-1/4} \right] \\
&> \left(\frac{16}{5\pi}\right)^{1/4} \left[1 - \frac{1}{2} \left(\frac{3\pi}{16}\right)^2 - 2^{-1/2} \right] > \frac{1}{10}. \quad \diamond
\end{aligned}$$

We now control the remaining integral of (48), namely $\int_{\varepsilon^r}^\infty \exp\left[\frac{\varepsilon-y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, x) dx$.

For that we shall resort to integration by parts, so that we need to estimate $\frac{d\Phi}{dx}(\pi^2, x)$. This will be made by using Lemma 8.2.2. The following analogue of Lemma 8.2.8 will actually be sufficient.

Lemma 8.3.5 *For any $y > 0$ and $x \geq \varepsilon^r$, uniformly as $\varepsilon \searrow 0$ we have*

$$\left| \Phi(\pi^2, x) \right| + \left| \frac{d\Phi}{dx}(\pi^2, x) \right| = 1_{\{\varepsilon^r \leq x < 1\}} \mathcal{O}(x^{-1/4}) + 1_{\{x \geq 1\}} \mathcal{O}\left(x e^{-\sqrt{x/8}}\right).$$

Proof Lemma 8.2.2 provides the uniform estimate: $|\Phi(\pi^2, x)| = \mathcal{O}\left[x \times \left(\operatorname{ch} \sqrt{x/2}\right)^{-1/2}\right] = \mathcal{O}\left[x e^{-\sqrt{x/8}}\right]$ for large x . Moreover Lemma 8.3.1 ensures that $\Phi(\pi^2, x) = \mathcal{O}(x^{-1/4})$ for small x . This settles the case of Φ . Then using Lemma 8.2.2 again and (37), as in the proof of Lemma 8.2.8 we obtain:

$$\frac{d \log \Phi}{dx}(\pi^2, x) = \sqrt{-1} \frac{d(\varphi + \frac{1}{2} \tilde{\varphi})}{dx}(\pi^2, x) + \mathcal{O}(1).$$

Note that the present case $\chi = \pi^2$ is simpler than the case $\chi \approx -4\pi^2$ of Lemma 8.2.8, since now we have $a \geq \pi$, hence $\operatorname{ch} a \geq \operatorname{ch} \pi$, so that the value of b does not matter so much in

the estimates needed here. Using again $\frac{\partial a}{\partial x} = \frac{b/2}{a^2 + b^2}$, $\frac{\partial b}{\partial x} = \frac{a/2}{a^2 + b^2}$ and Lemma 7.4, we thus have

$$\frac{d\varphi}{dx}(\pi^2, x) = \frac{1}{2} \frac{d}{dx} [\operatorname{arctg}\left(\frac{b}{a}\right) - \varphi_a(b)] = \frac{\pi^2}{4(\pi^4 + x^2)} - \frac{a \operatorname{sh}(2a) - b \sin(2b)}{4(a^2 + b^2)(\operatorname{ch}(2a) - \cos(2b))} = \mathcal{O}(1).$$

Then by Lemma 8.2.1 (again as in the proof of Lemma 8.2.8) and (38) we have

$$\begin{aligned} \frac{d\tilde{\varphi}}{dx}(\pi^2, x) &= \frac{\pi^2}{\pi^4 + x^2} - \frac{d}{dx} \operatorname{arctg} \left[\frac{2(a \sin b - b \operatorname{sh} a)}{2(a \operatorname{sh} a + b \sin b) - (a^2 + b^2)(\operatorname{ch} a + \cos b)} \right] \\ &= \frac{\pi^2}{\pi^4 + x^2} + \frac{\mathcal{O}((a^2 + b^2)^2(\operatorname{ch} a + \cos b)^2)}{(a^2 + b^2)^2(\operatorname{ch} a + \cos b)[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)]} \end{aligned}$$

which is bounded. The claim follows. \diamond

Now the analogue of Lemma 8.2.9 is the following.

Lemma 8.3.6 *We have $\int_{\varepsilon^r}^{\infty} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \times \Phi(\pi^2, x) dx = \mathcal{O}(\varepsilon^{2-r/4})$.*

Proof Integrating by parts, since $\Phi(\pi^2, \cdot)$ vanishes at infinity, using Lemma 8.3.5 we have :

$$\begin{aligned} &\int_{\varepsilon^r}^{\infty} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, x) dx \\ &= \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon - y} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \Phi(\pi^2, \varepsilon^r) + \frac{\varepsilon^2 \sqrt{-1}}{\varepsilon - y} \int_{\varepsilon^r}^{\infty} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \frac{d\Phi}{dx}(\pi^2, x) dx \\ &= \mathcal{O}(\varepsilon^{2-r/4}) + \mathcal{O}(\varepsilon^2) \int_{\varepsilon^r}^1 \mathcal{O}(x^{-1/4}) dx + \mathcal{O}(\varepsilon^2) \int_1^{\infty} \mathcal{O}(x e^{-\sqrt{x/8}}) dx = \mathcal{O}(\varepsilon^{2-r/4}). \quad \diamond \end{aligned}$$

We can now conclude this section, by the following exact equivalent of the oscillatory integral $\mathcal{I}_{\varepsilon}(y, 0)$ arising in the case $y > 0 = z$.

Proposition 8.3.7 *For $y > 0$, as $\varepsilon \searrow 0$ (with σ' as in Lemma 8.3.4) we have*

$$\Re \left\{ \int_0^{\infty} \exp\left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x\right] \times \Phi(\pi^2, x) dx \right\} = \frac{2^{3/4} \pi^{11/4} \sigma'}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times \frac{\varepsilon^{3/2}}{y^{3/4}} (1 + \mathcal{O}(\varepsilon^{2/5}))$$

and

$$\mathcal{I}_{\varepsilon}(y, 0) = \exp\left[-\pi^2 \frac{y - \varepsilon}{\varepsilon^2}\right] \times \frac{\varepsilon^{3/2}}{y^{3/4}} \times \frac{(2\pi)^{11/4} \sigma'}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times (1 + \mathcal{O}(\varepsilon^{2/5})).$$

Proof The first claim follows directly from (48) with $r = \frac{2}{5}$, Lemma 8.3.6 and Proposition 8.3.3. With Proposition 8.3.2, it entails the second one. \diamond

Propositions 6.3.1 and 8.3.7 together give the following wanted small time equivalent, which is the content of Theorem 2.1(iii) when $z = 0$.

Corollary 8.3.8 *For $y > 0$, as $\varepsilon \searrow 0$ we have*

$$p_{\varepsilon}(0; (0, y, 0)) \sim \exp\left[-\pi^2 \frac{y - \varepsilon}{\varepsilon^2}\right] \times \frac{\varepsilon^{-5/2}}{y^{3/4}} \times \frac{(2\pi/\operatorname{sh} \pi)^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}}} \int_0^{\infty} \frac{\sin\left(\frac{3\pi}{16} + x\right)}{x^{1/4}} dx.$$

8.4 Third sub-case: $z \neq 0$

Recall that according to Proposition 8.1.2 we have

$$\mathcal{I}_\varepsilon(y, z) = 2 \Re \left\{ \int_0^\infty \exp \left[\frac{\varepsilon - y}{\varepsilon^2} \sqrt{-1} x - \frac{z^2}{2\varepsilon^3} \frac{\sqrt{-1} x}{f(0, x)} \right] \Phi(x) dx \right\}.$$

We shall proceed somehow as in the preceding case in Section 8.3, $f(x)$ being positive at the saddle point, and roughly try to substitute $\frac{y - \varepsilon}{\varepsilon^2} + \frac{z^2}{2\varepsilon^3 f(0, x)}$ for $\frac{y - \varepsilon}{\varepsilon^2}$. However the dependence with respect to the non-constant $f(0, x)$ demands additional care.

First, since $\chi f(\chi, 0) > 0$ by (38), Proposition 8.3.2 at once becomes the following.

Proposition 8.4.1 *For any $z \neq 0$, $y \in \mathbb{R}$ and small $\varepsilon > 0$ we have*

$$\mathcal{I}_\varepsilon(y, z) = 2 \Re \left\{ \int_0^\infty \exp \left(- \left[\frac{y - \varepsilon}{\varepsilon^2} + \frac{z^2}{2\varepsilon^3 f(\pi^2, x)} \right] (\sqrt{-1} x + \pi^2) \right) \times \Phi(\pi^2, x) dx \right\}.$$

Then we take $\frac{3}{2} < r < 3$, to be specified later, and cut the above integral into:

$$\frac{1}{2} \mathcal{I}_\varepsilon(y, z) = \Re \left\{ \int_0^{\varepsilon^r} \dots \right\} + \Re \left\{ \int_{\varepsilon^r}^\infty \dots \right\} \equiv J_0^\varepsilon(y, z) + J_\infty^\varepsilon(y, z). \quad (49)$$

We first obtain the following (analogue to Proposition 8.3.3) behaviour of the main contribution $J_0^\varepsilon(y, z)$ to $\mathcal{I}_\varepsilon(y, z)$.

Proposition 8.4.2 *For any $(y, z) \in \mathbb{R} \times \mathbb{R}^*$, set $C_\varepsilon(y, z) := \left[\frac{\pi z^2}{2(\pi - 2 \operatorname{th} \frac{\pi}{2}) \varepsilon^3} + \frac{y - \varepsilon}{\varepsilon^2} \right]$, $K_\varepsilon(y, z) := \exp \left(- \pi^2 C_\varepsilon(y, z) \right)$, and $C^2 := \frac{\pi(\pi \operatorname{ch} \pi - 3 \operatorname{sh} \pi + 2\pi)}{4(\pi \operatorname{ch} \frac{\pi}{2} - 2 \operatorname{sh} \frac{\pi}{2})^2} > 0$. Then as $\varepsilon \searrow 0$ we have*

$$J_0^\varepsilon(y, z) = K_\varepsilon(y, z) \times \left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y - \varepsilon}{\varepsilon^2} \right]^{-3/4} \times \frac{2^{3/4} \pi^{11/4} \sigma'}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times [1 + \mathcal{O}(\varepsilon^{2r-3} + \varepsilon^{(3-r)/4})].$$

Proof In the proof of Lemma 8.3.1 we saw that

$$\frac{\sqrt{-1} x}{f(0, x)} = \frac{\pi^3}{\pi - 2 \operatorname{th} \frac{\pi}{2}} \times \left[1 + (1 + \sqrt{-1}) \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{\pi(\pi - 2 \operatorname{th} \frac{\pi}{2})} u + \mathcal{O}(u^2) \right]$$

at $x = -\sqrt{-1} \pi^2 + 2(1 - \sqrt{-1}) \pi u + u^2$, so that (taking $h = 2(1 - \sqrt{-1}) \pi u + u^2 \implies u = \frac{1+\sqrt{-1}}{4\pi} (1 - \frac{\sqrt{-1} h}{8\pi^2} + \mathcal{O}(h^2)) h$):

$$\frac{(\pi^2 + \sqrt{-1} h)}{f(0, -\sqrt{-1} \pi^2 + h)} = \frac{\pi^3}{\pi - 2 \operatorname{th} \frac{\pi}{2}} \times \left[1 + \sqrt{-1} \frac{3\pi - 6 \operatorname{th} \frac{\pi}{2} - \pi \operatorname{th}^2 \frac{\pi}{2}}{2\pi^2(\pi - 2 \operatorname{th} \frac{\pi}{2})} h + \mathcal{O}(h^2) \right],$$

i.e., for small h :

$$\frac{1}{f(\pi^2, h)} = \frac{\pi}{\pi - 2 \operatorname{th} \frac{\pi}{2}} \times \left[1 - \frac{\sqrt{-1} (\operatorname{sh} \pi - \pi) h}{2\pi^2 (\pi - 2 \operatorname{th} \frac{\pi}{2}) \operatorname{ch}^2 \frac{\pi}{2}} + \mathcal{O}(h^2) \right].$$

Therefore we have:

$$\begin{aligned} J_0^\varepsilon(y, z) &\equiv \int_0^{\varepsilon^r} \exp\left(-\left[\frac{y-\varepsilon}{\varepsilon^2} + \frac{z^2}{2\varepsilon^3 f(\pi^2, x)}\right](\sqrt{-1}x + \pi^2)\right) \times \Phi(\pi^2, x) dx \\ &= \exp(-\pi^2 C_\varepsilon(y, z)) \int_0^{\varepsilon^r} \exp\left(-\left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]\sqrt{-1}x + \mathcal{O}(\varepsilon^{2r-3})\right) \Phi(\pi^2, x) dx \\ &= \frac{K_\varepsilon(y, z)}{\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}} \int_0^{\left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]\varepsilon^r} \exp[-\sqrt{-1}x + \mathcal{O}(\varepsilon^{2r-3})] \times \Phi\left(\pi^2, \left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]^{-1}x\right) dx. \end{aligned}$$

Recall that we took $\frac{3}{2} < r < 3$. Now applying Lemma 8.3.1 (as for Proposition 8.3.3) yields:

$$\begin{aligned} J_0^\varepsilon(y, z) &= K_\varepsilon(y, z) \times \left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]^{-3/4} \frac{2^{3/4} \pi^{11/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \\ &\quad \times [1 + \mathcal{O}(\varepsilon^{2r-3} + \varepsilon^r)] \int_0^{\varepsilon^r \left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]} \sin\left[x + \frac{3\pi}{16}\right] \times x^{-1/4} dx. \end{aligned}$$

And finally

$$\int_0^{\varepsilon^r \left[\frac{C^2 z^2}{\varepsilon^3} + \frac{y-\varepsilon}{\varepsilon^2}\right]} \sin\left[x + \frac{3\pi}{16}\right] \times x^{-1/4} dx = \sigma' + \mathcal{O}(\varepsilon^{(3-r)/4}) = [1 + \mathcal{O}(\varepsilon^{(3-r)/4})] \sigma'. \quad \diamond$$

To estimate the contribution $J_\infty^\varepsilon(y, z)$, we crucially need the following.

Lemma 8.4.3 *We have $\Re \left[\frac{\pi^2 + \sqrt{-1}x}{f(\pi^2, x)} \right] > \frac{\pi^2}{f(\pi^2, 0)}$, for any positive x .*

Proof From (38) we compute: for any real $\chi > -\pi^2$ (we actually need $\chi = \pi^2$),

$$\begin{aligned} \Re \left[\frac{\chi + \sqrt{-1}x}{f(\chi, x)} \right] &= \frac{(a^4 - b^4)(\operatorname{ch} a + \cos b) - 2(a^2 - 3b^2) a \operatorname{sh} a - 2(3a^2 - b^2) b \sin b}{(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)} \\ &= \frac{\chi(2a^2 - \chi)(\operatorname{ch} a + \cos b) + 2(2a^2 - 3\chi) a \operatorname{sh} a - 2(2a^2 + \chi) b \sin b}{(2a^2 - \chi)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)}, \end{aligned}$$

and then, as $f(\pi^2, 0) = 1 - \frac{2 \operatorname{sh} \pi}{\pi (\operatorname{ch} \pi + 1)} = 1 - \frac{2}{\pi} \operatorname{th} \frac{\pi}{2} =: 1 - \lambda$ and $a^2 - b^2 = \pi^2$, we have:

$$\Re \left[\frac{\pi^2 + \sqrt{-1}x}{f(\pi^2, x)} \right] - \frac{\pi^2}{f(\pi^2, 0)} = \frac{N}{D}$$

with $D = [(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)][1 - \frac{2}{\pi} \operatorname{th} \frac{\pi}{2}] > 0$ and

$$\frac{1}{2} N = \frac{1}{2} [\pi^2(a^2 + b^2)(\operatorname{ch} a + \cos b) + 2(2a^2 - 3\pi^2) a \operatorname{sh} a - 2(2a^2 + \pi^2) b \sin b][1 - \frac{2}{\pi} \operatorname{th} \frac{\pi}{2}]$$

$$\begin{aligned}
& -\frac{1}{2}[(a^2 + b^2)(\operatorname{ch} a + \cos b) - 4(a \operatorname{sh} a + b \sin b) + 4(\operatorname{ch} a - \cos b)]\pi^2 \\
& = [2(1 - \lambda)a^2 + (3\lambda - 1)\pi^2]a \operatorname{sh} a - [2(1 - \lambda)a^2 - (\lambda + 1)\pi^2]b \sin b \\
& \quad - \lambda\pi^2(a^2 - \pi^2/2)(\operatorname{ch} a + \cos b) - 2\pi^2(\operatorname{ch} a - \cos b) \\
& = [2(1 - \lambda)b^2 + (\lambda + 1)\pi^2]\sqrt{\pi^2 + b^2} \operatorname{sh}\sqrt{\pi^2 + b^2} - [2(1 - \lambda)b^2 - (3\lambda - 1)\pi^2]b \sin b \\
& \quad - \lambda\pi^2(b^2 + \pi^2/2)(\operatorname{ch}\sqrt{\pi^2 + b^2} + \cos b) - 2\pi^2(\operatorname{ch}\sqrt{\pi^2 + b^2} - \cos b) \\
& = [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2]\pi\sqrt{1 + \frac{b^2}{\pi^2}} \operatorname{sh}\left[\pi\sqrt{1 + \frac{b^2}{\pi^2}}\right] + [(3\lambda - 1)\pi^2 - 2(1 - \lambda)b^2]b \sin b \\
& \quad - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2 \operatorname{ch}\left[\pi\sqrt{1 + \frac{b^2}{\pi^2}}\right] - (\lambda\frac{\pi^2}{2} - 2 + \lambda b^2)\pi^2 \cos b \\
& = \left[\left(\frac{3}{2}(1 - \lambda) - \lambda\frac{\pi^2}{4}\right)\pi \operatorname{sh} \pi + \frac{1-\lambda}{2}\pi^2 \operatorname{ch} \pi - 2(1 - \lambda)\pi^2 + \lambda\frac{\pi^4}{4}\right]b^2 + \mathcal{O}(b^4) \approx 0.186b^2 + \mathcal{O}(b^4).
\end{aligned}$$

(i) Positivity of N for $0 < b \leq 1.16$: in this range we successively have :

$$\begin{aligned}
\frac{N}{2} & > [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2]\pi\left[1 + \frac{b^2}{2\pi^2} - \frac{b^4}{8\pi^4}\right] \operatorname{sh}\left[\pi\left[1 + \frac{b^2}{2\pi^2} - \frac{b^4}{8\pi^4}\right]\right] \\
& \quad + [(3\lambda - 1)\pi^2 - 2(1 - \lambda)b^2]b\left(b - \frac{b^3}{6} + \frac{b^5}{125}\right) \\
& - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2 \operatorname{ch}\left[\pi\left[1 + \frac{b^2}{2\pi^2} - \frac{b^4}{8\pi^4} + \frac{b^6}{16\pi^6}\right]\right] - (\lambda\frac{\pi^2}{2} - 2 + \lambda b^2)\pi^2\left(1 - \frac{b^2}{2} + \frac{b^4}{24} - \frac{b^6}{740}\right); \\
& \quad \operatorname{sh}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right] \geq \operatorname{sh} \pi + \frac{b^2 \operatorname{ch} \pi}{2\pi} + \frac{b^4(\pi \operatorname{sh} \pi - \operatorname{ch} \pi)}{8\pi^3} \\
& \left(\text{indeed, if } f_0(b) := \operatorname{sh}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right] - \operatorname{sh} \pi - \frac{b^2 \operatorname{ch} \pi}{2\pi} - \frac{b^4(\pi \operatorname{sh} \pi - \operatorname{ch} \pi)}{8\pi^3}, \quad f_1(b) := \frac{f'_0(\pi b)}{b - b^3/2}, \text{ then}\right. \\
& \quad f_2(b) := \frac{(1 - b^2/2)^2 f'_1(b)}{\pi b} = \operatorname{sh}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right]\left(1 - \frac{b^2}{2}\right)^3 - \operatorname{sh} \pi, \\
& \quad \frac{f'_2(b)}{b(1 - b^2/2)^2} = \pi \operatorname{ch}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right]\left(1 - \frac{b^2}{2}\right)^2 - 3 \operatorname{sh}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right] \text{ decreases on } [0, \sqrt{2}] \text{ and is} \\
& \text{positive near 0, whence } f_2 \text{ increases near 0 and then possibly decreases, whence the same} \\
& \text{for } f_1 \text{ and } f_0, \text{ and } f_0(1.2) > 0, \text{ whence finally } f_0 > 0 \text{ on }]0, 1.2[);
\end{aligned}$$

$$\operatorname{ch}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3} + \frac{b^6}{16\pi^5}\right] \leq \operatorname{ch} \pi + \frac{b^2 \operatorname{sh} \pi}{2\pi} + \frac{b^4(\pi \operatorname{ch} \pi - \operatorname{sh} \pi)}{8\pi^3} + \frac{b^6 \operatorname{ch} \pi}{36\pi^4}$$

(indeed, if $g_0(b) :=$ the left hand side minus the right hand side, $g_1(b) := \frac{g'_0(\pi b)}{b - b^3/2}$, then $g_2(b) := \frac{(1 - b^2/2)^2 g'_1(b)}{\pi b} = \operatorname{ch}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right]\left(1 - \frac{b^2}{2}\right)^3 - \frac{(6 + 4b^2 - b^4) \operatorname{ch} \pi}{6}$, $g_3(b) := \frac{g'_2(b)}{b(1 - b^2/2)^2} = \pi \operatorname{sh}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3} + \frac{b^6}{16\pi^5}\right]\left(1 - \frac{b^2}{2}\right)^2 - 3 \operatorname{ch}\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3} + \frac{b^6}{16\pi^5}\right] - \frac{4 \operatorname{ch} \pi}{3(1 - b^2/2)}$ decreases on $[0, \sqrt{2}]$, which easily entails $g_2(b) < 0$, whence then easily $g_0(b) < 0$ on $]0, 1.2[)$).

Whence

$$N/2 > [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2]\left[\pi + \frac{b^2}{2\pi} - \frac{b^4}{8\pi^3}\right]\left[\operatorname{sh} \pi + \frac{b^2 \operatorname{ch} \pi}{2\pi} + \frac{b^4(\pi \operatorname{sh} \pi - \operatorname{ch} \pi)}{8\pi^3}\right]$$

$$+[(3\lambda - 1)\pi^2 - 2(1 - \lambda)b^2]b(b - \frac{b^3}{6} + \frac{b^5}{125}) - (\lambda\frac{\pi^2}{2} - 2 + \lambda b^2)\pi^2(1 - \frac{b^2}{2} + \frac{b^4}{24} - \frac{b^6}{740}) \\ - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2\left[\text{ch } \pi + \frac{b^2 \text{sh } \pi}{2\pi} + \frac{b^4(\pi \text{ch } \pi - \text{sh } \pi)}{8\pi^3} + \frac{b^6 \text{ch } \pi}{36\pi^4}\right] = b^2 P(b^2),$$

where P is polynomial of degree 4, which has the same behaviour as $\frac{N}{2b^2}$ near 0, increases first and then decreases on $[0, 1.16]$, with a positive value at 1.16. Therefore it is positive on $]0, 1.16]$, and then N as well.

(ii) Positivity of N for $1.16 \leq b \leq 5$: on the one hand, in this range we have

$$[(3\lambda - 1)\pi^2 - 2(1 - \lambda)b^2]b \sin b - (\lambda\frac{\pi^2}{2} - 2 + \lambda b^2)\pi^2 \cos b > 0$$

(indeed, since $\frac{\pi}{3} < 1.16 < \frac{\pi}{2} < \sqrt{\frac{3\lambda-1}{2-2\lambda}}\pi < \pi < \frac{3\pi}{2} < 5 < 2\pi$, denoting the left hand side by $f(b)$ we see at once that f is positive on $[\frac{\pi}{2}, \sqrt{\frac{3\lambda-1}{2-2\lambda}}\pi] \cup [\pi, \frac{3\pi}{2}]$, and that

$f'(b) = [(3\lambda - 3 + \lambda\frac{\pi^2}{2})\pi^2 + (\lambda\pi^2 - 6 + 6\lambda)b^2] \sin b - (1 - \lambda)(\pi^2 + 2b^2)b \cos b$ is positive on $[\frac{\pi}{2}, \pi]$ and negative on $[\frac{3\pi}{2}, 5]$. Since $f(5) \approx 20 > 0$, this shows the wanted positivity on $[\frac{\pi}{2}, 5]$. Moreover for $1.16 \leq b \leq \frac{\pi}{2}$ we have $\cos b < \sin b/\sqrt{3}$ and then $f'(b) > Q(b) \sin b$, with $Q(b) := (3\lambda - 3 + \lambda\frac{\pi^2}{2})\pi^2 + (\lambda\pi^2 - 6 + 6\lambda)b^2 - (\frac{1-\lambda}{\sqrt{3}})(\pi^2 + 2b^2)b$, which is easily seen to satisfy $Q''(b) > 2$, and then to be increasing and positive). Whence on the other hand:

$$\begin{aligned} \frac{N}{2 \text{ch} \sqrt{\pi^2 + b^2}} &\geq [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2] \sqrt{\pi^2 + b^2} \text{th} \sqrt{\pi^2 + b^2} - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2 \\ &\geq [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2] \frac{\pi+b}{\sqrt{2}} \text{th } \pi - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2 \\ &\geq [(\lambda + 1)\pi^2 + 2(1 - \lambda)(1.16)^2] \frac{\pi+1.16}{\sqrt{2}} \text{th } \pi - (\lambda\frac{\pi^2}{2} + 2 + \lambda(1.16)^2)\pi^2 > 0. \end{aligned}$$

(iii) Positivity of N for $b \geq 5$: estimating $|\cos b|$ and $|\sin b|$ by 1, we get

$$\begin{aligned} \frac{N}{2 \text{ch} \sqrt{\pi^2 + b^2}} &\geq [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2] \frac{\pi + b}{\sqrt{2}} \text{th} \sqrt{\pi^2 + 25} \\ &\quad - (\lambda\frac{\pi^2}{2} + 2 + \lambda b^2)\pi^2 - \frac{(\lambda\frac{\pi^2}{2} - 2 + \lambda b^2)\pi^2 + 2(1 - \lambda)b^3}{\text{ch}[(\pi + b)/\sqrt{2}]} \\ &\geq \frac{\text{th} \sqrt{\pi^2 + 25}}{\sqrt{2}} [(\lambda + 1)\pi^2 + 2(1 - \lambda)b^2](\pi + b) - \lambda\pi^2 b^2 - 50 \\ &\geq \frac{\text{th} \sqrt{\pi^2 + 25}}{\sqrt{2}} [(\lambda + 1)\pi^2 + 50(1 - \lambda)](\pi + 5) - 25\lambda\pi^2 - 50 > 0. \quad \diamond \end{aligned}$$

To handle the remaining contribution $J_\infty^\varepsilon(y, z)$ in (49) we still need the following.

Lemma 8.4.4 *The equation $(a + \sqrt{-1}b) \text{ch}(a + \sqrt{-1}b) - 3 \text{sh}(a + \sqrt{-1}b) + 2(a + \sqrt{-1}b) = 0$ has no solution such that $b \geq 0$, $a = \sqrt{\pi^2 + b^2}$.*

Proof This equation is equivalent to

$$(a \operatorname{ch} a - 3 \operatorname{sh} a) \cos b - b \operatorname{sh} a \sin b + 2a = 0 = b \operatorname{ch} a \cos b + (a \operatorname{sh} a - 3 \operatorname{ch} a) \sin b + 2b,$$

i.e., to

$$\frac{\cos \sqrt{a^2 - \pi^2}}{2} = \frac{3a \operatorname{ch} a - (2a^2 - \pi^2) \operatorname{sh} a}{(a^2 + \frac{9-\pi^2}{2}) \operatorname{sh}(2a) - 3a \operatorname{ch}(2a)}; \quad \frac{\sin \sqrt{a^2 - \pi^2}}{2} = \frac{3\sqrt{a^2 - \pi^2} \operatorname{sh} a}{(a^2 + \frac{9-\pi^2}{2}) \operatorname{sh}(2a) - 3a \operatorname{ch}(2a)}.$$

Now for $a \geq \pi$: $a^2 \frac{d}{da} (3 \operatorname{ch} a - (2a - \frac{\pi^2}{a}) \operatorname{sh} a) = (a^2 - \pi^2) \operatorname{sh} a - (2a^2 - \pi^2) a \operatorname{ch} a$
 $< -(2a^3 - a^2 - \pi^2 a + \pi^2) \operatorname{sh} a \leq -\pi^3 \operatorname{sh} a < 0$, whence $3a \operatorname{ch} a - (2a^2 - \pi^2) \operatorname{sh} a \leq -3\pi/2 < 0$.

And since $(a^2 + \frac{9-\pi^2}{2}) \operatorname{sh}(2a) - 3a \operatorname{ch}(2a) = \frac{(a^2+3a+\frac{9-\pi^2}{2})e^{2a}}{2} \left[\frac{a^2-3a+\frac{9-\pi^2}{2}}{a^2+3a+\frac{9-\pi^2}{2}} - e^{-4a} \right]$
 $\geq \frac{(a^2+3a+\frac{9-\pi^2}{2})e^{2a}}{2} \left[\frac{\pi^2-3\pi+\frac{9-\pi^2}{2}}{\pi^2+3\pi+\frac{9-\pi^2}{2}} - e^{-4\pi} \right] > 0$, we must have $\cos \sqrt{a^2 - \pi^2} < 0$, hence $a > \pi \frac{\sqrt{5}}{2}$.

This implies

$$\frac{\sin \sqrt{a^2 - \pi^2}}{2} < \frac{3\sqrt{a^2 - \pi^2} e^{-a}}{a^2 + 3a + \frac{9-\pi^2}{2}} \left[\frac{\frac{5\pi^2}{4} - \frac{3\pi\sqrt{5}}{2} + \frac{9-\pi^2}{2}}{\frac{5\pi^2}{4} + \frac{3\pi\sqrt{5}}{2} + \frac{9-\pi^2}{2}} - e^{-2\pi\sqrt{5}} \right]^{-1} < 16.44 e^{-\pi \frac{\sqrt{5}}{2}} \frac{3\sqrt{a^2 - \pi^2}}{a^2 + 3a + \frac{9-\pi^2}{2}},$$

whence $\sin \sqrt{a^2 - \pi^2} < \frac{3\sqrt{a^2 - \pi^2}}{a^2 + 3a + \frac{9-\pi^2}{2}}$. Now the polynomial

$\tilde{P}(a) := a^4 + 6a^3 + (18 - 4\pi^2 - c^2)a^2 + 3(9 - \pi^2)a + (\frac{9-\pi^2}{2})^2 + \pi^2 c^2$ is positive for $a > \pi \frac{\sqrt{5}}{2}$ and $c \leq 5.8$ (actually, $\tilde{P}''(a) > 0$ and $\tilde{P}'(a) > 0$ in that range), so that we have $5.8 \times \sqrt{a^2 - \pi^2} < a^2 + 3a + \frac{9-\pi^2}{2}$, hence $\sin \sqrt{a^2 - \pi^2} < \frac{3\sqrt{a^2 - \pi^2}}{a^2 + 3a + \frac{9-\pi^2}{2}} < \frac{3}{5.8}$, hence $\sqrt{a^2 - \pi^2} > \pi - \arcsin(\frac{3}{5.8})$, and then $a > 4$. Finally $a > 4$ implies

$$|\cos \sqrt{a^2 - \pi^2}| < \frac{2(2a^2 - 3a - \pi^2)}{a^2 + 3a + \frac{9-\pi^2}{2}} e^{-4} \left[\frac{4 + \frac{9-\pi^2}{2}}{28 + \frac{9-\pi^2}{2}} - e^{-16} \right]^{-1} < 4 \times 0.142 < 0.57,$$

and similarly:

$$|\sin \sqrt{a^2 - \pi^2}| < \frac{3\sqrt{a^2 - \pi^2} e^{-4}}{a^2 + 3a + \frac{9-\pi^2}{2}} \left[\frac{4 + \frac{9-\pi^2}{2}}{28 + \frac{9-\pi^2}{2}} - e^{-16} \right]^{-1} < \frac{3}{5.8} \times 0.142 < 0.08,$$

which forbids any solution $a > 4$. \diamond

We finally have the following control of the remaining contribution $J_\infty^\varepsilon(y, z)$ in (49).

Lemma 8.4.5 *For any $(y, z) \in \mathbb{R} \times \mathbb{R}^*$ we have $J_\infty^\varepsilon(y, z) = J_0^\varepsilon(y, z) \times \mathcal{O}(\varepsilon^{\frac{3-r}{4}})$.*

Proof In the spirit of Lemma 8.3.6, we integrate by parts. For that, consider the derivative of the exponent:

$$F_\varepsilon(x) := \frac{z^2}{2\varepsilon^3} \times \left(\frac{\sqrt{-1}}{f(\pi^2, 0)} - \frac{\sqrt{-1}}{f(\pi^2, x)} + \frac{(\sqrt{-1}x + \pi^2) \frac{df}{dx}(\pi^2, x)}{f(\pi^2, x)^2} \right) - \sqrt{-1} C_\varepsilon(y, z),$$

so that for any large A we have:

$$\begin{aligned}
& \Re \left\{ \int_{\varepsilon^r}^A \exp \left(\frac{z^2}{2\varepsilon^3} \left[\frac{1}{f(\pi^2, 0)} - \frac{1}{f(\pi^2, x)} \right] (\sqrt{-1}x + \pi^2) - C_\varepsilon(y, z)\sqrt{-1}x \right) \times \Phi(\pi^2, x) dx \right\} \\
&= \exp \left(\frac{-z^2}{2\varepsilon^3} \left[\Re \left(\frac{\pi^2 + \sqrt{-1}A}{f(\pi^2, A)} \right) - \frac{\pi^2}{f(\pi^2, 0)} \right] \right) \times \mathcal{O} \left[\left| \frac{\Phi(\pi^2, A)}{F_\varepsilon(A)} \right| \right] \\
&+ \exp \left(\frac{-z^2}{2\varepsilon^3} \left[\Re \left(\frac{\pi^2 + \sqrt{-1}\varepsilon^r}{f(\pi^2, \varepsilon^r)} \right) - \frac{\pi^2}{f(\pi^2, 0)} \right] \right) \times \mathcal{O} \left[\left| \frac{\Phi(\pi^2, \varepsilon^r)}{F_\varepsilon(\varepsilon^r)} \right| \right] \\
&- \Re \left\{ \int_{\varepsilon^r}^A \exp \left(\frac{z^2}{2\varepsilon^3} \left[\frac{1}{f(\pi^2, 0)} - \frac{1}{f(\pi^2, x)} \right] (\sqrt{-1}x + \pi^2) - C_\varepsilon(y, z)\sqrt{-1}x \right) \times \frac{d}{dx} \left[\frac{\Phi(\pi^2, x)}{F_\varepsilon(x)} \right] dx \right\} \\
&= \frac{\mathcal{O}(Ae^{-\sqrt{A/8}})}{|F_\varepsilon(A)|} + \frac{\mathcal{O}(\varepsilon^{-r/4})}{|F_\varepsilon(\varepsilon^r)|} + \int_{\varepsilon^r}^A \mathcal{O} \left[\left| \frac{d}{dx} \left[\frac{\Phi(\pi^2, x)}{F_\varepsilon(x)} \right] \right| \right] dx, \tag{50}
\end{aligned}$$

by Lemmas 8.4.3 and 8.3.5. We have to estimate $|F_\varepsilon(x)|^{-1}$, for $x \geq \varepsilon^r$. Now, writing f for $f(\pi^2, \cdot)$, we have

$$\begin{aligned}
F_\varepsilon(x)^{-1} &= \frac{\mathcal{O}(\varepsilon^3) f(x)}{\frac{(\pi^2 + \sqrt{-1}x)f'(x)}{f(x)} + \sqrt{-1} \left[\frac{f(x)}{f(0)} - 1 \right] - 2\sqrt{-1}z^{-2}\varepsilon^3 C_\varepsilon(y, z)f(x)} \\
&= \frac{\mathcal{O}(\varepsilon^3) f(x)}{\frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} - 1 - 2z^{-2}(y - \varepsilon)\varepsilon f(x)} \quad (\text{where } \mathcal{O}(\varepsilon^3) \text{ does not depend on } x).
\end{aligned}$$

We shall verify below that F_ε cannot vanish on \mathbb{R}_+^* .

In the proof of Proposition 8.4.2 we saw that for small positive x we have

$$\frac{1}{f(x)} = \frac{\pi}{\pi - 2 \operatorname{th} \frac{\pi}{2}} \times \left[1 - \frac{\sqrt{-1} (\operatorname{sh} \pi - \pi) x}{2\pi^2 (\pi - 2 \operatorname{th} \frac{\pi}{2}) \operatorname{ch}^2 \frac{\pi}{2}} + \mathcal{O}(x^2) \right].$$

Thus in particular we have $\frac{f'(0)}{f(0)} = \frac{\sqrt{-1} (\operatorname{sh} \pi - \pi)}{2\pi^2 (\pi - 2 \operatorname{th} \frac{\pi}{2}) \operatorname{ch}^2 \frac{\pi}{2}}$ and then

$$\frac{\pi^2 f'(0)}{f(0)} - \frac{2\sqrt{-1}\varepsilon^3}{z^2} C_\varepsilon(y, z)f(0) = \frac{\sqrt{-1} (\operatorname{sh} \pi - \pi)}{2(\pi - 2 \operatorname{th} \frac{\pi}{2}) \operatorname{ch}^2 \frac{\pi}{2}} - \sqrt{-1} + \mathcal{O}(\varepsilon) = \frac{\sqrt{-1} (3 \operatorname{sh} \pi - 2\pi - \pi \operatorname{ch} \pi)}{(1 + \operatorname{ch} \pi)(\pi - 2 \operatorname{th} \frac{\pi}{2})} + \mathcal{O}(\varepsilon) \neq 0.$$

Hence $F_\varepsilon(0) \neq 0$ and $\frac{\mathcal{O}(\varepsilon^{-r/4})}{|F_\varepsilon(\varepsilon^r)|} = \mathcal{O}(\varepsilon^{3-r/4})$. Then by (38) we have $f(x) = 1 + \mathcal{O}(x^{-1/2})$

for large x , which entails that f is bounded on \mathbb{R}_+ , and also:

$$\begin{aligned}
f'(x) &= \frac{\sqrt{-1} (\operatorname{sh} \sqrt{\pi^2 + \sqrt{-1}x} - \sqrt{\pi^2 + \sqrt{-1}x})}{(\pi^2 + \sqrt{-1}x)^{3/2} (1 + \operatorname{ch} \sqrt{\pi^2 + \sqrt{-1}x})} = \frac{[b(3a^2 + b^2) + \sqrt{-1}a(a^2 - 3b^2)]}{(a^2 + b^2)^3 \times (\operatorname{ch} a + \cos b)^2} \times \\
&\times [(\operatorname{ch} a + \cos b)(\operatorname{sh} a + \sqrt{-1} \sin b) - (a + \sqrt{-1}b)(1 + \operatorname{ch} a \cos b - \sqrt{-1} \operatorname{sh} a \sin b)],
\end{aligned}$$

which shows that for large x we have $f'(x) = \mathcal{O}((a^2 + b^2)^{-3/2}) = \mathcal{O}(x^{-3/2})$.

Hence on the one hand for large x we have

$$\frac{\mathcal{O}(A e^{-\sqrt{A/8}})}{|F_\varepsilon(A)|} = \frac{\mathcal{O}(\varepsilon^3 A e^{-\sqrt{A/8}})}{1 + \mathcal{O}(A^{-1/2} + \varepsilon)}, \quad \text{and} \quad \frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} = \mathcal{O}(x^{-1/2}).$$

This shows that for any small enough ε F_ε cannot vanish for large x . On the other hand,

$$\begin{aligned} \frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} &= \frac{\text{sh}\sqrt{\pi^2 + \sqrt{-1}x} - \sqrt{\pi^2 + \sqrt{-1}x}}{(\sqrt{\pi^2 + \sqrt{-1}x} - 2\text{th}[\sqrt{\pi^2 + \sqrt{-1}x}/2])(1 + \text{ch}\sqrt{\pi^2 + \sqrt{-1}x})} \\ &= \frac{\text{sh}(a + \sqrt{-1}b) - (a + \sqrt{-1}b)}{(a + \sqrt{-1}b)(1 + \text{ch}(a + \sqrt{-1}b)) - 2\text{sh}(a + \sqrt{-1}b)} \end{aligned}$$

is equal to 1 if and only if

$$(a + \sqrt{-1}b)\text{ch}(a + \sqrt{-1}b) - 3\text{sh}(a + \sqrt{-1}b) + 2(a + \sqrt{-1}b) = 0,$$

which is forbidden by Lemma 8.4.4. This shows that F_ε cannot vanish on \mathbb{R}_+^* , and moreover,

that for some small positive ε_0 we have $\inf_{0 < \varepsilon \leq \varepsilon_0, x \geq 0} \left| \frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} - 1 - 2z^{-2}(y - \varepsilon)\varepsilon f(x) \right| > 0$.

So far, according to (49) and (50) (in which we let $A \rightarrow \infty$) and the above, we have:

$$\begin{aligned} \frac{J_\infty^\varepsilon(y, z)}{K_\varepsilon(y, z)} &= \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) + \int_{\varepsilon^r}^\infty \left| \frac{\mathcal{O}(\varepsilon^3) f(x) \frac{d\Phi}{dx}(\pi^2, x)}{\frac{(\pi^2 + \sqrt{-1}x)f'(x)}{f(x)} - \sqrt{-1} + \mathcal{O}(\varepsilon)} \right| dx \\ &\quad + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \Phi(\pi^2, x) \times \frac{d}{dx} \left[\frac{f(x)}{\frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} - 1 - 2z^{-2}(y - \varepsilon)\varepsilon f(x)} \right] \right| dx \\ &= \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \frac{d\Phi}{dx}(\pi^2, x) \right| dx + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \Phi(\pi^2, x) f'(x) \right| dx \\ &\quad + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \Phi(\pi^2, x) \times \frac{d}{dx} \left[\frac{(\pi^2 + \sqrt{-1}x)f'(x)}{\sqrt{-1}f(x)} - 1 - 2z^{-2}(y - \varepsilon)\varepsilon f(x) \right] \right| dx \\ &= \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \frac{d\Phi}{dx}(\pi^2, x) \right| dx + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \Phi(\pi^2, x) f'(x) \right| dx \\ &\quad + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^\infty \left| \Phi(\pi^2, x) \right| \times \left[|(\pi^2 + \sqrt{-1}x)f''(x)| + |(\pi^2 + \sqrt{-1}x)f'(x)|^2 \right] dx. \end{aligned}$$

Using Lemma 8.3.5 this yields

$$\begin{aligned} \frac{J_\infty^\varepsilon(y, z)}{K_\varepsilon(y, z)} &= \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) + \mathcal{O}(\varepsilon^3) \int_{\varepsilon^r}^1 \left[1 + |f'(x)| + |f''(x)| + |f'(x)|^2 \right] x^{-1/4} dx \\ &\quad + \mathcal{O}(\varepsilon^3) \int_1^\infty \left[1 + x|f''(x)| + x|f'(x)|^2 \right] x e^{-\sqrt{x/8}} dx. \end{aligned}$$

We already saw above that $f'(x) = \mathcal{O}(x^{-3/2})$ for large x . Lemma 8.1.3 shows that f' and f'' are continuous on \mathbb{R}_+ . It remains to control f'' on $[1, \infty[$. Now from the expression of f' displayed above we have

$$f''(x) = \frac{3(\operatorname{sh}\sqrt{\pi^2 + \sqrt{-1}x} - \sqrt{\pi^2 + \sqrt{-1}x})}{4(\pi^2 + \sqrt{-1}x)^{5/2}(1 + \operatorname{ch}\sqrt{\pi^2 + \sqrt{-1}x})} - \frac{\operatorname{ch}\sqrt{\pi^2 + \sqrt{-1}x} - 1}{2(\pi^2 + \sqrt{-1}x)^2(1 + \operatorname{ch}\sqrt{\pi^2 + \sqrt{-1}x})} \\ + \frac{(\operatorname{sh}\sqrt{\pi^2 + \sqrt{-1}x} - \sqrt{\pi^2 + \sqrt{-1}x}) \operatorname{sh}\sqrt{\pi^2 + \sqrt{-1}x}}{2(\pi^2 + \sqrt{-1}x)^2(1 + \operatorname{ch}\sqrt{\pi^2 + \sqrt{-1}x})^2} = \mathcal{O}(x^{-2}).$$

Therefore we finally obtain :

$$\frac{J_\infty^\varepsilon(y, z)}{K_\varepsilon(y, z)} = \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) + \mathcal{O}(\varepsilon^3) \left[\int_{\varepsilon^r}^1 x^{-1/4} dx + \int_1^\infty e^{-\sqrt{x}/8} dx \right] = \mathcal{O}(\varepsilon^{3-\frac{r}{4}}) = \frac{J_0^\varepsilon(y, z)}{K_\varepsilon(y, z)} \mathcal{O}(\varepsilon^{\frac{3-r}{4}})$$

since $J_0^\varepsilon(y, z) \asymp K_\varepsilon(y, z) \varepsilon^{9/4}$ by Proposition 8.4.2. \diamond

We can now conclude this last section, by the following exact equivalent of the oscillatory integral $\mathcal{I}_\varepsilon(y, z)$ arising in the case $z \neq 0$.

Proposition 8.4.6 *For any $(y, z) \in \mathbb{R} \times \mathbb{R}^*$, with $C_\varepsilon(y, z) := \left[\frac{\pi z^2}{2(\pi - 2 \operatorname{th} \frac{\pi}{2}) \varepsilon^3} + \frac{y - \varepsilon}{\varepsilon^2} \right]$ and σ' as in Lemma 8.3.4, as $\varepsilon \searrow 0$ we have*

$$\mathcal{I}_\varepsilon(y, z) = \exp[-\pi^2 C_\varepsilon(y, z)] \times C_\varepsilon(y, z)^{-3/4} \times \frac{(2\pi)^{11/4} \sigma'}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}} (\operatorname{sh} \pi)^{1/4}} \times (1 + \mathcal{O}(\varepsilon^{1/3})).$$

Proof The first claim follows directly from (49) with $r = \frac{5}{3}$, Proposition 8.4.2 and Lemma 8.4.5. With Proposition 8.4.1, this gives the second one. \diamond

Propositions 6.3.1 and 8.4.6 together give the following wanted small time equivalent, which is the content of Theorem 2.1(iii) when $z \neq 0$, and actually extends Corollary 8.3.8.

Corollary 8.4.7 *For any $(y, z) \in \mathbb{R} \times \mathbb{R}^*$, as $\varepsilon \searrow 0$ we have*

$$p_\varepsilon(0; (0, y, z)) \sim \frac{\exp[-\pi^2 C_\varepsilon(y, z)]}{\varepsilon^4 C_\varepsilon(y, z)^{3/4}} \times \frac{(2\pi/\operatorname{sh} \pi)^{1/4}}{\sqrt{\pi - 2 \operatorname{th} \frac{\pi}{2}}} \int_0^\infty \frac{\sin(\frac{3\pi}{16} + x)}{x^{1/4}} dx.$$

9 References

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